

Universal Thermodynamics of the One-Dimensional Attractive Hubbard Model

Song Cheng

A thesis submitted for the degree of
Doctor of Philosophy
of the Australian National University

May 2018

Declaration

I hereby declare that the work presented in this thesis is my own except where otherwise acknowledged. This work has not been previously submitted for any other degree at any university or other tertiary education institution.

Song Cheng

Date

Dedicated to my parents and grandparents.

Acknowledgements

In the first place, I wish to thank my supervisor, Xi-Wen Guan, not only for his excellent guidance and kindly advices, but also for his incredible eagerness and dedication towards physics which always motivates me during my research candidature. As a beginner of integrable models, it is from him that I learnt how to capture and extract the physics of interest from the complicated derivations. He encourages me to work independently in his special way — promising me a celebrated reference book if I can read through it. And thanks to this, I have won the book. I also wish to express great gratitude to my chair of panel, Murray Batchelor. Without his full support I cannot have an opportunity to complete this project during my stay at the Department of Theoretical Physics.

I would like to thank Vladimir Bazhanov, Vladimir Mangazeev, and David Ridout for their efforts devoted to organizing so many inspiring seminars. Many thanks to other staff and visiting scholars, Mukunda Das, Angela Foerster and Chaoqing Dai for the interesting discussions with them. Meanwhile, I thank the other research students and postdocs at the department, especially Brendan Wilson, Gary Bosnjak, Callan Cain, Tianshu Liu, and Imam Alam. I thank them for their support and company. Besides, I am very grateful to my scientific collaborators, Yi-Cong Yu and Yu-Zhu Jiang. Their great numerical techniques and physical insights make our collaboration fruitful and enjoyable.

At last, I want to thank my parents and grandparents, whose love and concern constantly supports me to overwhelm the difficulties in my research work. Although my mother knows nothing about integrable systems, I still owe a lot to her due to her encouragement, understanding, and occasional criticism. This thesis would be a special present to my grandfather who passed away in 2015 and can not witness my graduation. He taught me how to consider a problem in every respect. Hopefully I did not disappoint him and neither in the future.

Abstract

The one-dimensional (1D) Fermi-Hubbard model describing interacting fermions on a lattice provides a paradigm of many-body physics, including spin-charge separation, fractional excitations, Fulde-Ferrell-Larkin-Ovchinnikov (FFLO) pairing, Mott insulating phase and phase transitions. Very recently, ultracold atoms trapped in optical lattices offer promising opportunities to test such traditional concepts. However, most of these studies were particularly restricted to repulsive interaction. The 1D attractive Hubbard model is a notoriously difficult problem and rarely studied in the literature due to the complicated bound states of multi-particles.

In this thesis, using the thermodynamic Bethe ansatz equations, we establish the equations of state in the strong coupling regime, by virtue of which the thermodynamic properties are analytically and numerically studied from the Luttinger liquid phase to the quantum critical region. Our analytical results show good agreement with the numerics, and could be used in fitting experimental data. We further derive the uniform scaling relations at quantum criticality, and read off the critical exponents. In the partially polarized phase IV, we introduce two effective chemical potentials for the two Luttinger liquids, which help us to construct the additivity rules for thermodynamic properties, susceptibility and compressibility. The simple additivity rules demonstrate the macroscopic behavior of this phase, reminiscent of two ‘non-interacting Fermi liquids’, and thereby reveal a free Fermi liquid nature of the 1D attractive Hubbard model. Moreover, in this phase IV, we study the long-distance asymptotics of various correlation functions at zero temperature. The spatial oscillating behavior of pair correlation function and its singular peaks in momentum space theoretically confirm an analog of FFLO state in the 1D attractive Hubbard. Lastly we observe that in contrast to the continuum Fermi gases, the correlation critical exponents, thermodynamics, Luttinger parameter, and Wilson ratio of the attractive Hubbard model essentially depend on lattice interacting effect, all of which under the lattice-gas map can be reduced to the case of Fermi gas.

This thesis provides a precise understanding of the universal low energy physics of attractive fermions on a lattice, and benchmark physics for ultracold atom experiments.

Contents

Abstract	ix
1 Introduction	1
2 The Hubbard Model	5
2.1 Hamiltonian	5
2.2 Lieb-Wu Equations	7
2.3 Thermodynamic Bethe Ansatz Equations	15
2.3.1 Zero Temperature Limit	24
2.3.2 Ground State Phase Diagram	28
2.4 Conclusion	31
3 Low Temperature thermodynamics	33
3.1 Equations of State	33
3.1.1 Preparation	33
3.1.2 Treatment of $\kappa(k)$	34
3.1.3 Treatment of $\epsilon'_n(\Lambda)$	38
3.1.4 Summary for Equations of State	41
3.2 Thermodynamic Quantities	42
3.2.1 Derivatives with respect to μ	43
3.2.2 Derivatives with respect to B	48
3.2.3 Summary for Thermodynamic Quantities	51
3.3 Iterative Solution	52
3.4 Quantum Criticality	57
3.4.1 μ_{c1} : Phase I-V	59
3.4.2 μ_{c2} : Phase I-II	61
3.4.3 μ_{c3} : Phase II-III	63

3.4.4	μ_{c4} : Phase II-IV	65
3.4.5	μ_{c5} : Phase IV-V	70
3.4.6	Summary for Quantum Criticality	74
4	Free-Fermi Liquids and Additivity Rules in FFLO-like Phase	77
4.1	Luttinger Parameters	78
4.2	Additivity Rule	83
4.3	Wilson Ratio	88
4.4	Conclusion	91
5	FFLO Signature for the 1D Attractive Hubbard Model	93
5.1	Ground State and Finite Size Correction	94
5.2	Low-Lying Excitations and Dressed Charge Matrix	96
5.3	Asymptotic Behavior of Correlation Functions at Zero Temperature	101
5.4	Conclusion	106
6	Conclusion	107
A	Some Results for the Convolution of Symmetric Functions	109
B	Table of Useful Integrals	111
B.1	Symmetric Integration	111
B.2	Fourier Transformation	111
B.3	Useful Identities	111
B.4	Integrals Involving Bessel Functions	112
C	Properties of the Dressed Energies at $T = 0$	113
D	Wiener-Hopf Method	127
E	Finite-Size Correction to the Ground State	133
F	Finite-Size Correction to Three Types of Elementary Excitations	141
	Bibliography	149

Introduction

How to capture the essential features of many-body physics through a simple model is always of great importance in condensed matter physics. In this regard, the Hubbard model [1–4] has long provided an active area of research since it was put forward as an instance of a Mott insulator and later considered as a potential high- T_c superconductor [5–8]. The Hubbard model has thus become a prototypical strongly correlated system which provides rich many-body phenomena, such as a Mott transition, superconductivity, spin-charge separation and a Fulde-Ferrell-Larkin-Ovchinnikov (FFLO) state. However, the Hubbard model, as a simplification of interacting fermions on realistic lattices, can be analytically resolved in neither two-dimensions (2D) nor three-dimensions (3D). The one-dimensional (1D) case within a single band is integrable, satisfying the Yang-Baxter equation [9,10]. It was firstly solved by Lieb and Wu in terms of the nested Bethe ansatz [11] and later on by Shastry in terms of the quantum inverse scattering method [12]. More specifically, since Lieb and Wu’s seminal work, the 1D repulsive Hubbard model has been investigated in various aspects, including, but not restricted to, thermodynamic properties in the ground state [13–20], low-lying excitations [21–30], finite temperature thermodynamics [27,31–36], and correlation functions [37–66].

The thermodynamics of the 1D Hubbard model is accessible through two alternative approaches – the thermodynamic Bethe ansatz (TBA) equations [31] and the quantum transfer matrix [67,68]. The former is established on the so-called ‘string’ hypothesis and Yang-Yang grand canonical ensemble approach [69], whereas the latter stems from the lattice path integral formulations for the partition function [70,71]. In principle, the low-lying excitations can be constructed with the help of the TBA equations in the zero temperature limit and by the logarithm of the Lieb-Wu equations [27,29,30]. Despite these systematic approaches and other methods employed for the study of the ground state properties [11,13–20] and low-lying excitations [21–26,28], a complete understanding of the universal thermodynamics and quantum criticality of the 1D Hubbard model has not yet been achieved. The key reason for preventing the solution of this problem is the difficulty of finding a suitable generating function for the equation

of state at low temperatures.

On the other hand, the correlation functions are also extremely difficult to calculate directly using the Bethe wave function. For a 1D conformally invariant system, the critical exponents determining the power law decay of correlation functions are connected with finite-size corrections to the ground state energy [72–75]. The 1D repulsive Hubbard model is conformally invariant only in the vicinity of Fermi points. The conformal field theory (CFT) approach provides one method to obtain the asymptotics of correlation functions [76]. The low-lying excitations provide a practicable opportunity for an investigation of long distance asymptotics of correlation functions [37, 38], where the finite-size corrections are accessible through the Bethe ansatz method [39, 40]. However, difficulties involved in the actual calculations of correlation functions usually prevent full access to the many-body correlations [41–55].

The mechanism of Cooper pairing in the 1D attractive Hubbard model has attracted attention [56–58] due to the discovery of high-temperature superconductors. In particular, the FFLO-like pair correlation and spin correlations are consequently investigated by various methods, such as density-matrix renormalization group [59–63], quantum Monte Carlo [64, 65] and CFT [66]. As far as we know, the confirmation of FFLO state in 1D mainly relies on numerics, although the application of conformal field theory (CFT) has provided a definite answer for the FFLO signature of 1D attractive Fermi gas [107].

Recently, ultra cold atoms in optical lattices are believed to provide a deep insight into the many-body physics. The optical lattice can be prepared by superposition of counter-propagating laser beams. Through the interaction between the induced dipole moment of atoms and the electromagnetic field gradient, the cold atoms can be trapped within optical lattices. A remarkable advantage of this technique is easy manipulation, where the inter-site hopping strength and on-site interaction can be adjusted by tuning the laser intensity and by using Feshbach resonance, respectively.

A milestone experiment for the Hubbard model is the one displayed in [77], where the quantum phase transition from superfluid to Mott insulator in the Bose-Hubbard model is observed. Later on, the Fermi-Hubbard model is realized [78]. From then on, trapping cold atoms on optical lattices becomes a promising method to simulate the many-body physics of the Hubbard model [79–85]. In particular, ultracold atoms offer an ideal platform for testing results predicted from 1D exactly solvable models [84].

Instead of Fermi liquid theory, the low-lying excitations in 1D many-body system can be described by the Tomonaga-Luttinger Liquid (TLL) theory. Although the microscopic origins of two kinds liquids are different, their macroscopic behaviors are similar. In fact, it is understood that the macroscopic behaviour of 1D materials, such as the spin compound CuPzN [86] and the heavy fermion

material YbNi_4P_2 [87], demonstrates a type of 3D Fermi liquid behaviour [88].

The motivation of our focus on the 1D attractive Hubbard is multifold. Firstly, the 1D attractive Hubbard model plays an important role in understanding many-body phenomena such as superconductivity, BEC-BCS crossover and FFLO-like correlation [59–63], yet only relatively few publications touch upon it [13–15, 18, 20, 28–30, 33, 36, 56–58]. Secondly, one expects to find universal behaviour for this model, including thermodynamics, quantum criticality and Luttinger liquid properties. Thirdly, regarding the complicated FFLO state, it is highly desirable to obtain simple rules to describe the nature of a free-Fermi liquid in the attractive Hubbard model. Fourthly, although various numerical methods have been applied in the study of FFLO state in 1D, the theoretical confirmation of FFLO state in 1D attractive Hubbard model is still an open problem. Last, but not least, the interplay of this work with experiments with ultracold atoms may broaden our knowledge of many-body physics through 1D exactly solvable models.

The rest of this thesis is organized as follows, in Chapter 2, we review the introduction of the 1D Hubbard model, and the derivation of the Bethe ansatz (BA) equations as well as the TBA equations, and determine the ground state phase diagram. In Chapter 3, we derive the equation of state in the strong coupling regime, on the basis of which we study the thermodynamic properties at low temperature and their critical behavior in the quantum critical region. In Chapter 4, we figure out the Luttinger parameters, then reveal the free Fermi liquid nature of the FFLO phase through the simple additivity rules for the thermodynamics quantities, and demonstrate the compressibility Wilson ratio as a powerful tool in identifying the low temperature phase diagram. In Chapter 5 we study various correlation functions through the CFT approach, and observe the FFLO-like state in the 1D attractive Hubbard model. Chapter 6 is reserved for the conclusion. The results presented in this thesis have been published [66, 89, 90], and Chapters 2-4 are based on [89, 90] while Chapter 5 on [66].

The Hubbard Model

2.1 Hamiltonian

In the 1960s, John Hubbard, Martin Gutzwiller and Junjiro Kanamori independently put forward a model to describe the electronic interaction in solids in their seminal papers [1–4], which nowadays is called Hubbard model. In this section we demonstrate how to extract this model from sophisticated depictions of real solids.

In general, a crystal consisting of ions and electrons can be considered as a static lattice made of ions with moving electrons, due to the much larger mass of ion than that of electron. Therefore, each electron within this lattice feels a periodic potential arising from the ions and the Coulomb interaction with the other electrons. The hamiltonian is expressed as

$$H = \sum_{j=1}^N \left(\frac{\vec{p}_j^2}{2m} + V_L(\vec{x}_j) \right) + \sum_{1 \leq i < j \leq N} V_C(\vec{x}_i - \vec{x}_j), \quad (2.1.1)$$

where $V_L(\vec{x})$ and $V_C(\vec{x})$ respectively stand for the periodic potential and Coulomb interaction, and N is the total number of electrons. In order to derive the second quantized form, we choose the Wannier functions as the basis of states, which are defined by

$$W_{\alpha}(\vec{x}, \vec{r}_i) = \frac{1}{\sqrt{L}} \sum_{\vec{k}} \exp(-i\vec{k} \cdot \vec{r}_i) \psi_{\alpha, \vec{k}}(\vec{x}). \quad (2.1.2)$$

Here we have used the Bloch function

$$\psi_{\alpha, \vec{k}}(\vec{x}) = \exp(i\vec{k} \cdot \vec{x}) u_{\alpha, \vec{k}}(\vec{x}) \quad (2.1.3)$$

with band index α , wave vector \vec{k} and periodic function $u_{\alpha, \vec{k}}(\vec{x})$. As the eigenfunction of the one-particle hamiltonian $h = \frac{\vec{p}^2}{2m} + V_L(\vec{x})$, the Bloch function satisfies

the static Schrödinger Equation

$$h\Psi_{\alpha,\vec{k}}(\vec{x}) = e_{\alpha,\vec{k}}\Psi_{\alpha,\vec{k}}(\vec{x}). \quad (2.1.4)$$

We define the annihilation operators in Wannier and Bloch representations as $c_{\alpha,i,a}$ and $c_{\alpha,\vec{k},a}$ respectively, which comply with the following transformation

$$c_{\alpha,i,a} = \frac{1}{\sqrt{L}} \sum_{\vec{k}} \exp(i\vec{k} \cdot \vec{r}_i) c_{\alpha,\vec{k},a}. \quad (2.1.5)$$

The relation between their creation operators can be derived by simple conjugation of the last equation. Hereby the field operator now reads

$$\Psi_a(\vec{x}) = \sum_{\alpha,\vec{k}} \Psi_{\alpha,\vec{k}}(\vec{x}) c_{\alpha,\vec{k},a} = \sum_{\alpha,i} W_{\alpha}(\vec{x}, \vec{r}_i) c_{\alpha,i,a}. \quad (2.1.6)$$

By virtue of the last equation, we can recast the hamiltonian eq. (2.1.1) into

$$H = \sum_{\alpha,i,j,a} T_{ij}^{\alpha} c_{\alpha,i,a}^{\dagger} c_{\alpha,j,a} + \frac{1}{2} \sum_{\alpha,\beta,\gamma,\delta} \sum_{i,j,k,l} \sum_{a,b} U_{ijkl}^{\alpha\beta\gamma\delta} c_{\alpha,i,a}^{\dagger} c_{\beta,j,b}^{\dagger} c_{\gamma,k,b} c_{\delta,l,a} \quad (2.1.7)$$

where the coefficients of the hopping term and that of the interaction term are respectively expressed as

$$T_{ij}^{\alpha} = \frac{1}{L} \sum_{\vec{k}} \exp(i\vec{k} \cdot (\vec{r}_i - \vec{r}_j)) e_{\alpha,\vec{k}}, \quad (2.1.8)$$

$$U_{ijkl}^{\alpha\beta\gamma\delta} = \int d\vec{x} d\vec{x}' W_{\alpha}^*(\vec{x}, \vec{r}_i) W_{\beta}^*(\vec{x}', \vec{r}_j) V_C(\vec{x} - \vec{x}') W_{\alpha}(\vec{x}, \vec{r}_l) W_{\alpha}(\vec{x}', \vec{r}_m). \quad (2.1.9)$$

We notice that $U_{iiii}^{\alpha\beta\gamma\delta}$ represents the interaction among electrons at the same lattice site, which is much more important than the other coefficients $U_{ijkl}^{\alpha\beta\gamma\delta}$ in narrow band systems. Furthermore, we just consider the single band case, i.e., $\alpha = 1$, then the on-site interaction $U_{iiii}^{\alpha\beta\gamma\delta}$ is much simplified as U , and thus the hamiltonian in eq. (2.1.7) can be approximated as

$$H = \sum_{i,j} \sum_{a=\uparrow,\downarrow} T_{ij} c_{i,a}^{\dagger} c_{j,a} + \frac{U}{2} \sum_i \sum_{a,b=\uparrow,\downarrow} c_{i,a}^{\dagger} c_{i,b}^{\dagger} c_{i,b} c_{i,a}, \quad (2.1.10)$$

We then introduce operator $n_{i\uparrow} = c_{i\uparrow}^{\dagger} c_{i\uparrow}$ ($n_{i\downarrow} = c_{i\downarrow}^{\dagger} c_{i\downarrow}$) to denote the number of spin-up (spin-down) electrons at the same site j , and thus obtain

$$H = \sum_{i,j} \sum_{a=\uparrow,\downarrow} T_{ij} c_{i,a}^{\dagger} c_{j,a} + U \sum_i n_{i\uparrow} n_{i\downarrow}, \quad (2.1.11)$$

where we have considered the Pauli exclusion principle and the anti-commutation relation $\{c_{i,\sigma}, c_{j,\sigma'}^\dagger\} = \delta_{i,j} \delta_{\sigma,\sigma'}$.

If we only keep the hopping terms between nearest neighbours, and assume that the hopping coefficient is a constant $-t$, we finally arrive at the second quantized Hubbard model hamiltonian,

$$H = -t \sum_{j=1}^L \sum_{a=\uparrow,\downarrow} c_{j,a}^\dagger c_{j+1,a} + c_{j+1,a}^\dagger c_{j,a} + U \sum_{j=1}^L n_{j\uparrow} n_{j\downarrow}. \quad (2.1.12)$$

As a remark, here we did not consider the terms with coefficients T_{ii} , because they can be absorbed into the chemical potential in a grand canonical ensemble, which is exactly the ensemble applied in our later research on universal thermodynamics.

2.2 Lieb-Wu Equations

The one-dimensional (1D) Hubbard model was firstly resolved by E. Lieb and F. Y. Wu in 1968 through the application of the Bethe ansatz [11]. This technique can trace back to the wave function suggested by H. Bethe in 1931 for the 1D Heisenberg chain [100], by virtue of which he successfully transformed the diagonalization of the hamiltonian to solving a set of coupled algebraic equations. However, Bethe's idea did not receive much attention until Lieb and Liniger's work on the 1D Bose gas with δ -function interaction in 1963 [101]. Later on, C. N. Yang [9] and M. Gaudin [102] generalized this Bethe ansatz to the the model with internal degrees of freedom in their investigation into the 1D $SU(2)$ Fermi gas. Lieb and Wu noticed the similarity between the two-body scattering matrices of the Fermi gas and Hubbard model, and thus derived the following Lieb-Wu equations

$$e^{ik_j L} = \prod_{\alpha=1}^M \frac{\sin k_j - \Lambda_\alpha + iU/4}{\sin k_j - \Lambda_\alpha - iU/4}, \quad j = 1, 2, 3, \dots, N; \quad (2.2.1)$$

$$\prod_{j=1}^N \frac{\sin k_j - \Lambda_\beta + iU/4}{\sin k_j - \Lambda_\beta - iU/4} = - \prod_{\alpha=1}^M \frac{\Lambda_\alpha - \Lambda_\beta + iU/2}{\Lambda_\alpha - \Lambda_\beta - iU/2}, \quad \beta = 1, 2, 3, \dots, M; \quad (2.2.2)$$

where N , L , M are the number of particles, lattice sites and spin-down particles, respectively.

The rest of this section will be devoted to a brief review of the derivation of the above Lieb-Wu equations.

The eigenstate of the hamiltonian in eq. (2.1.7) is built by

$$|\Psi\rangle = \sum_{x_1 \dots x_N} \sum_{\sigma_1 \dots \sigma_N} \Psi(x_1, \sigma_1; x_2, \sigma_2; \dots; x_N, \sigma_N) c_{x_1, \sigma_1}^\dagger c_{x_2, \sigma_2}^\dagger \dots c_{x_N, \sigma_N}^\dagger |0\rangle, \quad (2.2.3)$$

where $1 \leq x_i \leq L$ ($i = 1, \dots, N$) is the electron position, $\sigma_j = \uparrow, \downarrow$ stand for spins, and $|0\rangle$ represents the vacuum state. Without losing generality, we assume the first N_\downarrow electrons are spin-down and the rest spin-up. In light of the Pauli exclusion principle, the wave function complies with $\delta_{x_j, x_i} \Psi(x_1, \sigma_1; x_2, \sigma_2; \dots; x_N, \sigma_N) = 0$ when $i, j \leq N_\downarrow$ or $i, j \geq N_\downarrow + 1$.

Substituting the above eigenstate into the static Schrödinger equation $H|\Psi\rangle = E|\Psi\rangle$ yields

$$-\sum_j [f(x_1, \dots, x_j + 1, \dots, x_N) + f(x_1, \dots, x_j - 1, \dots, x_N)] + U \sum_{k < k'} \delta_{x_k, x_{k'}} f(x_1, \dots, x_N) = E f(x_1, \dots, x_N), \quad (2.2.4)$$

where for simplicity we let $f(x_1, \dots, x_N) = \Psi(x_1, \sigma_1; \dots; x_N, \sigma_N)$.

According to the Bethe ansatz, the wave function is expressed as

$$f(x_1, \dots, x_j + 1, \dots, x_N) = \sum_{Q, P} \theta_H(x_{Q_1} < x_{Q_2} < \dots < x_{Q_N}) A(Q, P) \exp \left(i \sum_{j=1}^N k_{P_j} x_{Q_j} \right), \quad (2.2.5)$$

where $\theta_H(x < y < z)$ is the Heaviside step function, $A(Q, P)$ is the coefficient, both Q and P are permutations of $(1, 2, \dots, N)$, and summation is carried out over all possible permutations.

Assuming that there is no double occupation, inserting the above wave function into eq. (2.2.4), one can obtain the eigenenergy

$$E = -2 \sum_{j=1}^N \cos k_j. \quad (2.2.6)$$

We choose two permutations, $Q^{(1)} = (Q_1, Q_2, Q_3, \dots, Q_N)$ and $Q^{(2)} = (Q_1, Q_3, Q_2, \dots, Q_N)$ which define two regions

$$\begin{aligned} R^{(1)} : & 0 < x_{Q_1} < x_{Q_2} < x_{Q_3} < x_{Q_4} < \dots < x_{Q_N} < N, \\ R^{(2)} : & 0 < x_{Q_1} < x_{Q_3} < x_{Q_2} < x_{Q_4} < \dots < x_{Q_N} < N. \end{aligned} \quad (2.2.7)$$

The wave functions in these regions are denoted as $f_1(x_1, x_2, \dots, x_N)$ and $f_2(x_1, x_2, \dots, x_N)$. If there is one double occupation $x_{Q_2} = x_{Q_3} = x$, then to be continuous in the boundary of neighbouring regions the wave function should satisfy

$$f_1(x_1, \dots, x_{Q_2} = x, \dots, x_{Q_3} = x, \dots, x_N) = f_2(x_1, \dots, x_{Q_3} = x, \dots, x_{Q_2} = x, \dots, x_N), \quad (2.2.8)$$

which together with the Bethe ansatz wave function in eq. (2.2.5) results in

$$A(Q^{(1)}, P^{(1)}) + A(Q^{(1)}, P^{(2)}) = A(Q^{(2)}, P^{(1)}) + A(Q^{(2)}, P^{(2)}). \quad (2.2.9)$$

Here we have denoted two new permutations $P^{(1)} = (P_1, P_2, P_3, P_4 \dots, P_N)$ and $P^{(2)} = (P_1, P_3, P_2, P_4, \dots, P_N)$.

Moreover, under this double occupation condition that $x_{Q_2} = x_{Q_3} = x$, the static Schrödinger equation (2.2.4) can be recast into

$$\begin{aligned} & -f_2(x_1, \dots, x+1, \dots, x, \dots, x_N) - f_1(x_1, \dots, x-1, \dots, x, \dots, x_N) \\ & -f_1(x_1, \dots, x, \dots, x+1, \dots, x_N) - f_2(x_1, \dots, x, \dots, x-1, \dots, x_N) \\ & - \sum_{j \neq Q_2, Q_3}^N [f_1(x_1, \dots, x_j+1, \dots, x_N) + f_1(x_1, \dots, x_j-1, \dots, x_N)] \\ & + U f_1(x_1, \dots, x, \dots, x, \dots, x_N) = E f_1(x_1, \dots, x, \dots, x, \dots, x_N). \end{aligned} \quad (2.2.10)$$

In the region $R^{(1)}$, provided that there is no double occupation, eq. (2.2.4) can be rewritten as

$$- \sum_j^N [f_1(x_1, \dots, x_j+1, \dots, x_N) + f_1(x_1, \dots, x_j-1, \dots, x_N)] = E f_1(x_1, \dots, x_N), \quad (2.2.11)$$

to which applying $x_{Q_2}, x_{Q_3} \rightarrow x$ results in

$$\begin{aligned} & -f_1(x_1, \dots, x+1, \dots, x, \dots, x_N) - f_1(x_1, \dots, x, \dots, x-1, \dots, x_N) \\ & -f_1(x_1, \dots, x, \dots, x+1, \dots, x_N) - f_1(x_1, \dots, x-1, \dots, x, \dots, x_N) \\ & - \sum_{j \neq Q_2, Q_3} [f_1(x_1, \dots, x_j+1, \dots, x_N) + f_1(x_1, \dots, x_j-1, \dots, x_N)] = E f_1(x_1, \dots, x, \dots, x_N). \end{aligned} \quad (2.2.12)$$

Subtracting eq. (2.2.12) from eq. (2.2.10) leads to

$$\begin{aligned} & f_1(x_1, \dots, x+1, \dots, x, \dots, x_N) - f_2(x_1, \dots, x+1, \dots, x, \dots, x_N) \\ & + f_1(x_1, \dots, x, \dots, x-1, \dots, x_N) - f_2(x_1, \dots, x, \dots, x-1, \dots, x_N) \\ & + U f_1(x_1, \dots, x, \dots, x, \dots, x_N) = 0, \end{aligned} \quad (2.2.13)$$

inserting Bethe wave function into which results in

$$\begin{aligned} & [A(Q^{(1)}, P^{(1)}) - A(Q^{(2)}, P^{(2)})] (e^{ik_{P_2}} + e^{-ik_{P_3}}) + [A(Q^{(1)}, P^{(2)}) - A(Q^{(2)}, P^{(1)})] (e^{ik_{P_3}} + e^{-ik_{P_2}}) \\ & + U [A(Q^{(1)}, P^{(1)}) + A(Q^{(1)}, P^{(2)})] = 0. \end{aligned} \quad (2.2.14)$$

The last equation and eq. (2.2.9) give rise to

$$A(Q^{(1)}, P^{(1)}) = Y_{P_3 P_2}^{23} A(Q^{(1)}, P^{(2)}), \quad (2.2.15)$$

where $Y_{P_3 P_2}^{23}$ is expressed by

$$Y_{P_3 P_2}^{23} = \frac{(\sin k_{P_2} - \sin k_{P_3}) \hat{P}_{Q_2 Q_3} + iU/2}{\sin k_{P_2} - \sin k_{P_3} - iU/2}. \quad (2.2.16)$$

Here the operator $\hat{P}_{Q_2 Q_3}$ is defined by $\hat{P}_{Q_2 Q_3} A(Q_1, Q_2, Q_3, \dots, Q_N; P) = A(Q_1, Q_3, Q_2, \dots, Q_N; P)$.

We write down the general case of eq. (2.2.15)

$$A(Q; \dots, P_a, P_b, \dots) = Y_{P_b P_a}^{ab} A(Q; \dots, P_b, P_a, \dots), \quad (2.2.17)$$

with

$$Y_{P_b P_a}^{ab} = \frac{(\sin k_{P_a} - \sin k_{P_b}) \hat{P}_{Q_a Q_b} + iU/2}{\sin k_{P_a} - \sin k_{P_b} - iU/2}. \quad (2.2.18)$$

Now we add the spin indices to the coefficient $A(Q, P)$ and introduce spin exchange operator $\hat{P}_{\sigma_1 \sigma_2} = \frac{1}{2}(1 + \vec{\sigma}_1 \cdot \vec{\sigma}_2)$, and then the exchange antisymmetry of the wave function requires that

$$\begin{aligned} & A_{\sigma_{Q_1} \dots \sigma_{Q_a} \dots \sigma_{Q_b} \dots \sigma_{Q_N}}(\dots, Q_a, Q_b, \dots; \dots, P_b, P_a, \dots) \\ &= -\hat{P}_{\sigma_{Q_a} \sigma_{Q_b}} A_{\sigma_{Q_1} \dots \sigma_{Q_a} \dots \sigma_{Q_b} \dots \sigma_{Q_N}}(\dots, Q_b, Q_a, \dots; \dots, P_b, P_a, \dots), \end{aligned} \quad (2.2.19)$$

which together with eq. (2.2.17) results in

$$A_{\sigma_1 \dots \sigma_N}(\dots, Q_a, Q_b, \dots; \dots, P_a, P_b, \dots) = S_{k_{P_b} k_{P_a}} A_{\sigma_1 \dots \sigma_N}(\dots, Q_b, Q_a, \dots; \dots, P_b, P_a, \dots) \quad (2.2.20)$$

with two body scattering matrix

$$S_{k_{P_b} k_{P_a}} = \frac{\sin k_{P_b} - \sin k_{P_a} + i\frac{U}{2} \hat{P}_{\sigma_{Q_a} \sigma_{Q_b}}}{\sin k_{P_b} - \sin k_{P_a} - i\frac{U}{2}}. \quad (2.2.21)$$

Taking account of the periodic boundary condition, without losing generality one can confirm that

$$\Psi(x_1 = 0, \sigma_1; x_2, \sigma_2; \dots; x_N, \sigma_N) = \Psi(x_1 = L, \sigma_1; x_2, \sigma_2; \dots; x_N, \sigma_N). \quad (2.2.22)$$

Substituting the Bethe ansatz eq. (2.2.5) into the both sides of last equation leads

to

L.H.S.

$$= \sum_{Q^{(\alpha)}, P} \theta_H \left(x_1 = x_{Q_1^{(\alpha)}} = 0 < x_{Q_2^{(\alpha)}} < \cdots < x_{Q_N^{(\alpha)}} \right) A \left(Q^{(\alpha)}, P \right) \exp \left(i \sum_{j=1}^N k_{P_j} x_{Q_j^{(\alpha)}} \right), \quad (2.2.23)$$

where $Q^{(\alpha)}$ stand for those permutations of which '1' lies in the first position, i.e. $Q^{(\alpha)} = (Q_1^{(\alpha)} = 1, Q_2^{(\alpha)}, \dots, Q_N^{(\alpha)})$;

R.H.S.

$$\begin{aligned} &= \sum_{Q^{(\beta)}, P'} \theta_H \left(x_{Q_1^{(\beta)}} < x_{Q_2^{(\beta)}} < \cdots < x_{Q_N^{(\beta)}} = x_1 = L \right) A \left(Q^{(\beta)}, P' \right) \exp \left(i \sum_{j=1}^{N-1} k_{P'_j} x_{Q_j^{(\beta)}} + i k_{P'_N} L \right) \\ &= \sum_{Q^{(\alpha)}, P'} \theta_H \left(x_{Q_2^{(\alpha)}} < x_{Q_3^{(\alpha)}} < \cdots < x_{Q_N^{(\alpha)}} < x_{Q_1^{(\alpha)}} = x_1 = L \right) A \left(Q^{(\beta)}, P' \right) \exp \left(i \sum_{j=2}^N k_{P_j} x_{Q_j^{(\alpha)}} + i k_{P_1} L \right) \end{aligned} \quad (2.2.24)$$

where we let $Q^{(\beta)} = (Q_1^{(\beta)}, Q_2^{(\beta)}, \dots, Q_N^{(\beta)} = 1) = (Q_2^{(\alpha)}, Q_3^{(\alpha)}, \dots, Q_N^{(\alpha)}, Q_1^{(\alpha)} = 1)$, $P = (P_1, P_2, \dots, P_N)$ and $P' = (P_2, P_3, \dots, P_N, P_1)$.

Now equality $R.H.S. = L.H.S.$ follows that

$$A \left(Q_1^{(\alpha)} = 1, \dots, Q_N^{(\alpha)}; P_1, \dots, P_N \right) = A \left(Q_2^{(\alpha)}, \dots, Q_N^{(\alpha)}, Q_1^{(\alpha)} = 1; P_2, \dots, P_N, P_1 \right) \exp(i k_{P_1} L), \quad (2.2.25)$$

which together with eq. (2.2.20) yields

$$S_{j+1,j} \cdots S_{N,j} S_{1,j} \cdots S_{j-1,j} = \exp(i k_j L), \quad j = 1, 2, \dots, N. \quad (2.2.26)$$

We define two functions, $b(x) = -x/(-x+i)$ and $c(x) = i/(-x+i)$. It is easy to see that, $b(x) + c(x) = 1$, $b(0) = 0$, and $c(0) = 1$. With them, we can rewrite the scattering matrix $S_{i,j}$ as

$$S_{i,j} = b \left(\frac{\sin k_j - \sin k_i}{U/2} \right) + c \left(\frac{\sin k_j - \sin k_i}{U/2} \right) \hat{P}_{\sigma_i \sigma_j}. \quad (2.2.27)$$

We introduce Lax operator

$$L_j(u) = b(u - u_j) + c(u - u_j) \hat{P}_{\sigma_j \tau}, \quad (2.2.28)$$

where τ is Pauli matrix in auxiliary space C_2 . For simplicity we denote $u_j =$

$\sin k_j/(U/2)$. It is obvious that $L_j(u_j) = \hat{P}_{\sigma_j \tau}$. By virtue of the Lax operator, we define the monodromy matrix and transfer matrix as

$$T_N(u) = L_1(u) L_2(u) \cdots L_N(u), \quad (2.2.29)$$

$$t(u) = \text{tr}_{C_2} T_N(u). \quad (2.2.30)$$

One can find that $L_l(u_j) \hat{P}_{\sigma_j \tau} = \hat{P}_{\sigma_j \tau} S_{l,j}$, and thus

$$\text{tr}_{C_2} T_N(u_j) = S_{j+1,j} \cdots S_{N,j} S_{1,j} \cdots S_{j-1,j}, \quad (2.2.31)$$

which transforms eq. (2.2.26) into

$$\text{tr}_{C_2} T_N(u_j) = \exp(ik_j L), \quad j = 1, 2, \dots, N. \quad (2.2.32)$$

We rewrite the Lax operator as

$$L_j(u) = b(u - u_j) \mathbb{I} \otimes \mathbb{I}_j + \frac{1}{2} c(u - u_j) (\mathbb{I} \otimes \mathbb{I}_j + \tau^1 \otimes \sigma_j^1 + \tau^2 \otimes \sigma_j^2 + \tau^3 \otimes \sigma_j^3), \quad (2.2.33)$$

where σ_j^β and τ^β ($\beta = 1, 2, 3$) are the Pauli matrices of (x, y, z) directions in Hilbert space \mathcal{H}_j and auxiliary space C_2 ; \mathbb{I}_j and \mathbb{I} are the identity matrices in \mathcal{H}_j and C_2 .

We further introduce the $\hat{R}(u)$ matrix in $C_2 \otimes C_2$

$$\hat{R}(u) = c(u) \mathbb{I} \otimes \mathbb{I} + \frac{1}{2} b(u) (\mathbb{I} \otimes \mathbb{I} + \tau^1 \otimes \tau^1 + \tau^2 \otimes \tau^2 + \tau^3 \otimes \tau^3), \quad (2.2.34)$$

and then one can find that it satisfies

$$\hat{R}(u - v) [L_j(u) \otimes L_j(v)] = [L_j(v) \otimes L_j(u)] \hat{R}(u - v), \quad (2.2.35)$$

$$\hat{R}(u - v) [T_N(u) \otimes T_N(v)] = [T_N(v) \otimes T_N(u)] \hat{R}(u - v), \quad (2.2.36)$$

where the direct product of Lax operators (transfer matrices) with different spectral parameters is only carried out over the auxiliary spaces.

Equations (2.2.35) and (2.2.36) are called the Yang-Baxter relations, which imply

$$[\text{tr}_{C_2} T_N(u), \text{tr}_{C_2} T_N(v)] = 0. \quad (2.2.37)$$

Thereby one can simultaneously diagonalize eq. (2.2.32).

Provided that the monodromy matrix living in auxiliary space C_2 is expressed

in the form

$$T_N(u) = \begin{bmatrix} A(u) & B(u) \\ C(u) & D(u) \end{bmatrix}, \quad (2.2.38)$$

then substituting it into eq. (2.2.36) results in

$$[B(u), B(v)] = 0, \quad (2.2.39)$$

$$\begin{aligned} A(u)B(v) &= \frac{1}{b(v-u)}B(v)A(u) - \frac{c(v-u)}{b(v-u)}B(u)A(v), \\ D(u)B(v) &= \frac{1}{b(u-v)}B(v)D(u) - \frac{c(u-v)}{b(u-v)}B(u)D(v). \end{aligned} \quad (2.2.40)$$

Here $A(u)$, $B(u)$, $C(u)$ and $D(u)$ are operators of $\mathcal{H}_1 \otimes \mathcal{H}_2 \dots \otimes \mathcal{H}_N$.

We assume a reference state of the system $|0\rangle_{\text{ref}} = |\uparrow_1, \uparrow_2, \dots, \uparrow_N\rangle$, and the Lax pair acting on it follows

$$L_j(u)|0\rangle_{\text{ref}} = \begin{bmatrix} 1 & c(u-u_j)\sigma_j^- \\ 0 & b(u-u_j) \end{bmatrix} |0\rangle_{\text{ref}}, \quad (2.2.41)$$

which indicates that

$$T_N(u)|0\rangle_{\text{ref}} = \begin{bmatrix} 1 & B(u) \\ 0 & \prod_{j=1}^N b(u-u_j) \end{bmatrix} |0\rangle_{\text{ref}}. \quad (2.2.42)$$

Explicitly one can obtain

$$\begin{aligned} A(u)|0\rangle_{\text{ref}} &= |0\rangle_{\text{ref}} \\ B(u)|0\rangle_{\text{ref}} &= \sum_{j=1}^N \dots |\uparrow, \dots, \downarrow, \dots, \uparrow\rangle \\ C(u)|0\rangle_{\text{ref}} &= 0 \\ D(u)|0\rangle_{\text{ref}} &= \prod_{j=1}^N b(u-u_j)|0\rangle_{\text{ref}}. \end{aligned} \quad (2.2.43)$$

Here $B(u)$ acting on the reference state gives rise to a linear superposition of N states, each of which has a flipped spin. Hence one can consider $B(u)$ as a one-spin flip operator. We use $B(u)$ to construct eigenstate $B(v_1)B(v_2)\dots B(v_M)|0\rangle_{\text{ref}}$ where M spins have been flipped. Let the transfer matrix act on it, then

$$\begin{aligned} & t(u)B(v_1)B(v_2)\dots B(v_M)|0\rangle_{\text{ref}} \\ &= [A(u) + B(u)]B(v_1)B(v_2)\dots B(v_M)|0\rangle_{\text{ref}} \end{aligned}$$

$$\begin{aligned}
&= \left[\frac{1}{\prod_{i=1}^M b(v_i - u)} + \frac{\prod_{j=1}^N b(u - u_j)}{\prod_{i=1}^M b(u - u_j)} \right] B(v_1) \cdots B(v_M) |0\rangle_{\text{ref}} \\
&\quad - \sum_{j=1}^M \frac{c(v_j - u)}{b(v_j - u)} \left[\frac{1}{\prod_{\alpha=1, \alpha \neq j}^M b(v_\alpha - v_j)} - \frac{\prod_{i=1}^N b(v_j - u_j)}{\prod_{\alpha=1, \alpha \neq j}^M b(v_j - v_\alpha)} \right] B(v_1) \cdots B(v_{j-1}) B(u) B(v_{j+1}) \cdots B(v_M) |0\rangle_{\text{ref}} \\
&= \Lambda(u) B(v_1) \cdots B(v_M) |0\rangle_{\text{ref}} + \text{unwanted terms.} \tag{2.2.44}
\end{aligned}$$

If $B(v_1) \cdots B(v_M) |0\rangle_{\text{ref}}$ and $\Lambda(u)$ are the eigenstate and eigenvalue of transfer matrix $t(u)$ respectively, then eq. (2.2.32) is rewritten as

$$\prod_{i=1}^M \frac{1}{b(v_i - u_j)} = e^{ik_j L}. \tag{2.2.45}$$

To eliminate the unwanted terms, one can derive that

$$\prod_{j=1}^N b(v_i - u_j) = \prod_{\alpha}^M \frac{b(v_i - v_\alpha)}{b(v_\alpha - v_i)}. \tag{2.2.46}$$

We introduce notation $v_\alpha = \Lambda_\alpha / (U/2) + i/2$ and substitute $u_j = \sin k_j / (U/2)$ into eqs. (2.2.45) and (2.2.46), and then we arrive at the Lieb-Wu equations shown in eqs. (2.2.1) and (2.2.2).

From then on, we will use the following Hubbard hamiltonian

$$H = - \sum_{j=1}^L \sum_{a=\uparrow, \downarrow} \left(c_{j,a}^\dagger c_{j+1,a} + c_{j+1,a}^\dagger c_{j,a} \right) + u \sum_{j=1}^L (1 - 2n_{j,\uparrow}) (1 - 2n_{j,\downarrow}), \tag{2.2.47}$$

where

$$u = \frac{U}{4t}$$

with $t = 1$ represents the interaction between particles, $u > 0$ for repulsion and $u < 0$ for attraction. Note that this interaction u has nothing to do with the spectral parameter u in the Lax operator eq. (2.2.28). Consequently the Lieb-Wu equations are rewritten as

$$e^{ik_j L} = \prod_{\alpha=1}^M \frac{\sin k_j - \Lambda_\alpha + iu}{\sin k_j - \Lambda_\alpha - iu}, \quad j = 1, 2, 3, \dots, N; \tag{2.2.48}$$

$$\prod_{j=1}^N \frac{\sin k_j - \Lambda_\beta + iu}{\sin k_j - \Lambda_\beta - iu} = - \prod_{\alpha=1}^M \frac{\Lambda_\alpha - \Lambda_\beta + 2iu}{\Lambda_\alpha - \Lambda_\beta - 2iu}, \quad \beta = 1, 2, 3, \dots, M. \tag{2.2.49}$$

The eigenenergy and total momentum are respectively given by

$$E = -2 \sum_{j=1}^N \cos k_j + u(L - 2N), \quad (2.2.50)$$

$$P = \left[\sum_{j=1}^N k_j \right] \mod 2\pi. \quad (2.2.51)$$

2.3 Thermodynamic Bethe Ansatz Equations

As a set of coupled nonlinear algebraic equations, the Bethe ansatz equations are actually impossible to resolve under the thermodynamic limit, which blocks our way to the universal thermodynamics of the many-body system of interest. To overcome this difficulty, C. N. Yang and C. P. Yang developed a grand canonical ensemble method in the study of the 1D δ interaction Bose gas, wherein the Bethe ansatz equations are transformed into a set of coupled nonlinear integral equations called the thermodynamic Bethe ansatz (TBA) equations [69]. With respect to the 1D repulsive Hubbard model, M. Takahashi proposed the so-called string hypothesis to describe the root pattern of the Lieb-Wu equations, and then derive the TBA equations following the Yang-Yang approach [31]. The TBA equations with attractive interaction is derived by Lee and Schlottmann [33], and by Essler and Korepin [30].

The TBA equations allow us an opportunity to precisely determine the universal thermodynamics of a many-body system. Their solution are the so-called dressed energies, which physically account for the elementary excitations. In this section, we demonstrate the derivation from Lieb-Wu equations to TBA equations for the 1D attractive Hubbard model.

Under the thermodynamic limit, i.e., $N, M, L \rightarrow \infty$ with finite $\frac{N}{L}$ and $\frac{M}{L}$, the roots of the Lieb-Wu equations in the complex plane could be categorized into three classes,

- single real k 's;
- the α -th k - Λ string of length m , for which there are $2m$ k 's,

$$\begin{aligned} k_\alpha^1 &= \arcsin(\Lambda_\alpha'^m + im|u|), \\ k_\alpha^2 &= \arcsin(\Lambda_\alpha'^m + i(m-2)|u|), \\ k_\alpha^3 &= \pi - k_\alpha^2, \\ &\vdots \\ k_\alpha^{2m-2} &= \arcsin(\Lambda_\alpha'^m - i(m-2)|u|), \end{aligned} \quad (2.3.1)$$

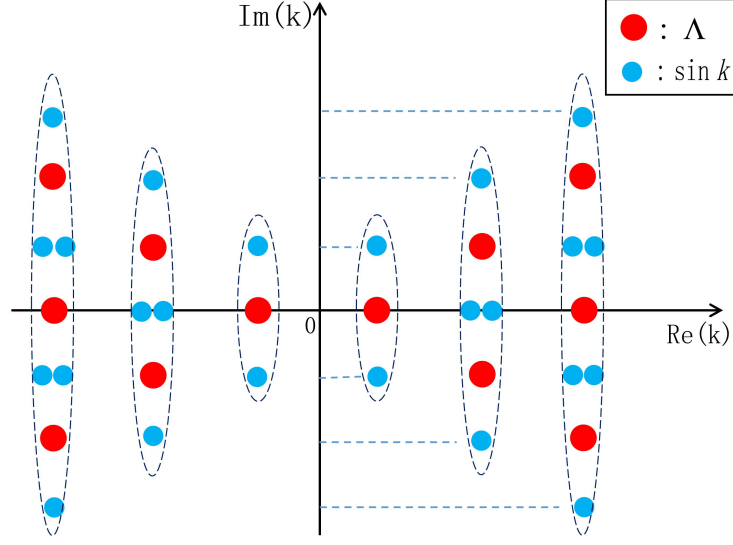


Figure 2.1: A schematic configuration of the k - Λ strings of length-1,2,3. The k - Λ bound states are formed by the charge momenta and spin rapidities displayed within the dashed boundaries. In each k - Λ bound state $\sin k$'s share a real part with the spin rapidities. A length- m k - Λ string contains m rapidities in Λ -space and $2m$ quasimomenta in k -space.

$$k_{\alpha}^{2m} = \arcsin(\Lambda_{\alpha}^{'m} - im|u|),$$

accompanied by m spin-rapidities,

$$\Lambda_{\alpha}^{'m,j} = \Lambda_{\alpha}^{'m} + i(m+1-2j)|u|, \quad j = 1, 2, 3, \dots, m, \quad (2.3.2)$$

in Λ space, where $\Lambda_{\alpha}^{'m}$ is the real center of this k - Λ string;

- the α -th Λ - Λ string of length m ,

$$\Lambda_{\alpha}^{m,j} = \Lambda_{\alpha}^m + i(m+1-2j)|u|, \quad j = 1, 2, 3, \dots, m. \quad (2.3.3)$$

where Λ_{α}^m is the real center of this Λ string. The Λ - Λ strings represent the spin wave bound states in the spin sector.

Now we let M_m , M'_m , and \mathcal{M}_e denote the number of Λ strings of length m , of k - Λ strings of length m , and of single real k 's, respectively. It is easy to see that

$$M = \sum_{m=1}^{\infty} m(M_m + M'_m), \quad (2.3.4)$$

$$N = \mathcal{M}_e + \sum_{m=1}^{\infty} 2mM'_m. \quad (2.3.5)$$

Substituting the string hypothesis into the Lieb-Wu equations and taking logarithms leads to the discrete nested BA equations

$$k_j L = 2\pi I_j + \sum_{m=1}^{\infty} \sum_{\alpha=1}^{M'_m} \theta\left(\frac{\sin k_j - \Lambda'_\alpha{}^m}{m|u|}\right) + \sum_{m=1}^{\infty} \sum_{\alpha=1}^{M_m} \theta\left(\frac{\sin k_j - \Lambda_\alpha^m}{m|u|}\right), \quad (2.3.6)$$

$$\sum_{j=1}^{N-2M'} \theta\left(\frac{\Lambda_\alpha^n - \sin k_j}{n|u|}\right) = 2\pi J_\alpha^n + \sum_{m=1}^{\infty} \sum_{\beta=1}^{M_m} \Theta_{nm}\left(\frac{\Lambda_\alpha^n - \Lambda_\beta^m}{n|u|}\right), \quad (2.3.7)$$

$$\begin{aligned} 2L \operatorname{Re} [\arcsin(\Lambda'_\alpha{}^n + i n |u|)] &= 2\pi J'_\alpha{}^n + \sum_{j=1}^{N-2M'} \theta\left(\frac{\Lambda'_\alpha{}^n - \sin k_j}{n|u|}\right) \\ &+ \sum_{m=1}^{\infty} \sum_{\beta=1}^{M'_m} \Theta_{nm}\left(\frac{\Lambda'_\alpha{}^n - \Lambda'_\beta{}^m}{|u|}\right), \end{aligned} \quad (2.3.8)$$

where $M' = \sum_{m=1}^{\infty} m M'_m$ is the total number of Λ 's involved in the k - Λ strings, $\theta(x) = 2 \arctan(x)$, and

$$\Theta_{nm}(x) = \begin{cases} \theta\left(\frac{x}{|n-m|}\right) + 2\theta\left(\frac{x}{|n-m|+2}\right) + \cdots + 2\theta\left(\frac{x}{n+m-2}\right) + \theta\left(\frac{x}{n+m}\right) & \text{if } n \neq m \\ 2\theta\left(\frac{x}{2}\right) + 2\theta\left(\frac{x}{4}\right) + \cdots + 2\theta\left(\frac{x}{2n-2}\right) + \theta\left(\frac{x}{2n}\right) & \text{if } n = m. \end{cases}$$

The quantum numbers I_j , J_α^n and $J'_\alpha{}^n$ are either integers or half-odd integers, stemming from the multivaluedness of the log functions. They are determined by the relations

$$\begin{aligned} I_j &= \begin{cases} \text{integers} & \text{if } \sum_{m=1}^{\infty} (M'_m + M_m) \text{ is even} \\ \text{half-odd integers} & \text{if } \sum_{m=1}^{\infty} (M'_m + M_m) \text{ is odd,} \end{cases} \\ J_\alpha^n &= \begin{cases} \text{integers} & \text{if } N - M_n \text{ is odd} \\ \text{half-odd integers} & \text{if } N - M_n \text{ is even,} \end{cases} \\ J'_\alpha{}^n &= \begin{cases} \text{integers} & \text{if } N - M'_n \text{ is odd} \\ \text{half-odd integers} & \text{if } N - M'_n \text{ is even.} \end{cases} \end{aligned}$$

Substituting the string hypotheses eqs. (2.3.1) to (2.3.3) into eqs. (2.2.50) and (2.2.51) results in

$$E = -2 \sum_{j=1}^{N-2M'} \cos k_j - 4 \sum_{n=1}^{\infty} \sum_{\alpha=1}^{M'_n} \operatorname{Re} \left[\sqrt{1 - (\Lambda'_\alpha{}^n + i n |u|)^2} \right] - 2uN + uL \quad (2.3.9)$$

and

$$P = \left[\sum_{j=1}^{N-2M'} k_j + \sum_{n=1}^{\infty} \sum_{\alpha=1}^{M'_n} (2\text{Re}(\arcsin(\Lambda'_\alpha{}^n + i n|u|)) + (n-1)\pi) \right] \mod 2\pi. \quad (2.3.10)$$

We define counting functions for the quantum numbers

$$y_L(k_j) = \frac{2\pi I_j}{L}, \quad z_{n,L}(\Lambda_\alpha^n) = \frac{2\pi J_\alpha^n}{L}, \quad z'_{n,L}(\Lambda'_\alpha{}^n) = \frac{2\pi J'_\alpha{}^n}{L}, \quad (2.3.11)$$

which under the thermodynamic limit read $y(k_j)$, $z_n(\Lambda_\alpha^n)$ and $z'_n(\Lambda'_\alpha{}^n)$ respectively. Due to the relation between counting functions and quantum numbers, those counting functions satisfy

$$2\pi [\rho^p(k) + \rho^h(k)] = \frac{dy(k)}{dk}, \quad (2.3.12)$$

$$2\pi [\sigma_n^p(\Lambda) + \sigma_n^h(\Lambda)] = \frac{dz_n(\Lambda)}{d\Lambda}, \quad (2.3.13)$$

$$2\pi [\sigma'_n{}^p(\Lambda) + \sigma'_n{}^h(\Lambda)] = \frac{dz'_n(\Lambda)}{d\Lambda}, \quad (2.3.14)$$

where ρ^p , σ_n^p , $\sigma'_n{}^p$ (ρ^h , σ_n^h , $\sigma'_n{}^h$) are root densities of particles (holes) in quasi-momenta of excess fermions, Λ - Λ string parameter space and k - Λ string space, respectively. Then one could derive that

$$\rho^p(k) + \rho^h(k) = \frac{1}{2\pi} - \cos k \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} d\Lambda a_n(\sin k - \Lambda) [\sigma_n^p(\Lambda) + \sigma'_n{}^p(\Lambda)], \quad (2.3.15)$$

$$\sigma_n^h(\Lambda) = \int_{-\pi}^{\pi} dk a_n(\sin k - \Lambda) \rho^p(k) - \sum_{m=1}^{\infty} A_{nm} * \sigma_m^p \Big|_{\Lambda}, \quad (2.3.16)$$

$$\sigma'_n{}^h(\Lambda) = \frac{1}{\pi} \text{Re} \left[\frac{1}{\sqrt{1 - (\Lambda + i n|u|)^2}} \right] - \sum_{m=1}^{\infty} A_{nm} * \sigma'_m{}^p \Big|_{\Lambda} - \int_{-\pi}^{\pi} dk a_n(\sin k - \Lambda) \rho^p(k), \quad (2.3.17)$$

where $*$ represents convolution, and new function $a_n(x)$ is defined by

$$a_n(x) = \frac{1}{2\pi} \frac{2n|u|}{(nu)^2 + x^2}, \quad (2.3.18)$$

with

$$A_{nm} * f|_x = \delta_{n,m} \cdot f(x) + \int_{-\infty}^{\infty} \frac{dy}{2\pi} \frac{d}{dx} \Theta_{nm} \left(\frac{x-y}{|u|} \right) f(y). \quad (2.3.19)$$

Here we denoted the derivatives of the function Θ_{nm} as

$$\begin{aligned} & \frac{1}{2\pi} \frac{d}{dx} \Theta_{nm} \left(\frac{x-y}{|u|} \right) \\ &= \begin{cases} a_{|n-m|}(x-y) + 2a_{|n-m|+2}(x-y) + \dots + 2a_{n+m-2}(x-y) + a_{n+m}(x-y) & \text{if } n \neq m, \\ 2a_2(x-y) + 2a_4(x-y) + \dots + 2a_{2n-2}(x-y) + a_{2n}(x-y) & \text{if } n = m. \end{cases} \end{aligned} \quad (2.3.20)$$

The root distribution functions eqs. (2.3.15) to (2.3.17) determine spin and charge excitations, spin dynamics and full energy spectra. In the grand canonical ensemble, the Gibbs free energy per site can be expressed in terms of these root densities

$$f(\mu, B, T) = e - \mu \cdot n_c - 2B \cdot m - T \cdot s \quad (2.3.21)$$

$$\begin{aligned} &= \int_{-\pi}^{\pi} dk (-2 \cos k - \mu - 2u - B) \rho^p(k) \\ &\quad - \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} d\Lambda \sigma_n'^p(\Lambda) \left[4 \operatorname{Re} \sqrt{1 - (\Lambda_\alpha'^n + i n |u|)^2} + n(2\mu + 4u) \right] \\ &\quad + \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} d\Lambda 2n \cdot B \cdot \sigma_n^p(\Lambda) - T \cdot s + u, \end{aligned} \quad (2.3.22)$$

where μ is the chemical potential, B the magnetic field and T the temperature. In the above equations n_c is the particle density, $m = \frac{N-2M}{2L}$ the magnetization and s the entropy per site.

Following the Yang-Yang grand canonical description [69], the entropy per site is explicitly given by

$$\begin{aligned} s &= \int_{-\pi}^{\pi} dk \left\{ \mathcal{L}[\rho^p(k) + \rho^h(k)] - \mathcal{L}[\rho^p(k) - \rho^h(k)] \right\} \\ &\quad + \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} d\Lambda \left\{ \mathcal{L}[\sigma_n'^p(\Lambda) - \sigma_n'^h(\Lambda)] - \mathcal{L}[\sigma_n'^p(\Lambda)] - \sigma_n'^h(\Lambda) \right\} \\ &\quad + \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} d\Lambda \left\{ \mathcal{L}[\sigma_n^p(\Lambda) - \sigma_n^h(\Lambda)] - \mathcal{L}[\sigma_n^p(\Lambda)] - \mathcal{L}[\sigma_n^h(\Lambda)] \right\}, \end{aligned} \quad (2.3.23)$$

with $\mathcal{L}[f(x)] = f(x) \ln[f(x)]$. In the following, we only consider the physics with $B \geq 0$ and $\mu \leq 0$.

As a functional of particle (hole) densities $\rho^{p(h)}$, $\sigma^{p(h)}$, $\sigma'^{p(h)}$, the free energy must be stationary with respect to variations of the particle (hole) densities in

the state of thermodynamic equilibrium . Therefore, one could derive that

$$0 = \delta f = \int_{-\pi}^{\pi} dk \left(\frac{\delta f}{\delta \rho^p} \delta \rho^p + \frac{\delta f}{\delta \rho^h} \delta \rho^h \right) + \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} d\Lambda \left(\frac{\delta f}{\delta \sigma_n^p} \delta \sigma_n^p + \frac{\delta f}{\delta \sigma_n^h} \delta \sigma_n^h \right) + \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} d\Lambda \left(\frac{\delta f}{\delta \sigma_n'^p} \delta \sigma_n'^p + \frac{\delta f}{\delta \sigma_n'^h} \delta \sigma_n'^h \right), \quad (2.3.24)$$

with

$$\frac{\delta f}{\delta \rho^p} = -2 \cos k - \mu - 2u - B - T \ln \left(1 + \frac{\rho^h}{\rho^p} \right), \quad (2.3.25)$$

$$\frac{\delta f}{\delta \rho^h} = -T \ln \left(1 + \frac{\rho^p}{\rho^h} \right), \quad (2.3.26)$$

$$\frac{\delta f}{\delta \sigma_n^p} = -4 \operatorname{Re} \sqrt{1 - (\Lambda + i n |u|)^2} - n(2\mu + 4u) - T \ln \left(1 + \frac{\sigma_n'^h}{\sigma_n^p} \right), \quad (2.3.27)$$

$$\frac{\delta f}{\delta \sigma_n^h} = -T \ln \left(1 + \frac{\sigma_n'^p}{\sigma_n^h} \right), \quad (2.3.28)$$

$$\frac{\delta f}{\delta \sigma_n^p} = 2nB - T \ln \left(1 + \frac{\sigma_n^h}{\sigma_n^p} \right), \quad (2.3.29)$$

$$\frac{\delta f}{\delta \sigma_n^h} = -T \ln \left(1 + \frac{\sigma_n^p}{\sigma_n^h} \right), \quad (2.3.30)$$

where we have used the expression for entropy per site, eq. (2.3.23).

In addition, from eqs. (2.3.15) to (2.3.17), one could derive

$$\delta \rho^p + \delta \rho^h = \frac{1}{2\pi} - \cos k \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} d\Lambda a_n (\sin k - \Lambda) \cdot [\delta \sigma_n^p + \delta \sigma_n'^p], \quad (2.3.31)$$

$$\delta \sigma_n^h = \int_{-\pi}^{\pi} dk a_n (\sin k - \Lambda) \delta \rho^p - \sum_{m=1}^{\infty} A_{nm} * \delta \sigma_m^p, \quad (2.3.32)$$

$$\delta \sigma_n'^h = - \int_{-\pi}^{\pi} dk a_n (\sin k - \Lambda) \delta \rho^p - \sum_{m=1}^{\infty} A_{nm} * \delta \sigma_m'^p. \quad (2.3.33)$$

Substituting eqs. (2.3.25) to (2.3.33) into eq. (2.3.24), and setting the coefficients of $\delta \rho^p$, $\delta \sigma^p$ and σ'^p to zero respectively gives rise to the results

$$\ln \zeta(k) = \frac{-2 \cos k - \mu - 2u - B}{T} + \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} d\Lambda a_n (\sin k - \Lambda) \ln \left(1 + \frac{1}{\eta_n'(\Lambda)} \right) - \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} d\Lambda a_n (\sin k - \Lambda) \ln \left(1 + \frac{1}{\eta_n(\Lambda)} \right), \quad (2.3.34)$$

$$\ln(1 + \eta_n(\Lambda)) = \frac{2nB}{T} + \int_{-\pi}^{\pi} dk \cos k a_n(\sin k - \Lambda) \ln \left(1 + \frac{1}{\zeta(k)} \right) + \sum_{m=1}^{\infty} A_{nm} * \ln \left(1 + \frac{1}{\eta_m} \right) \Big|_{\Lambda}, \quad (2.3.35)$$

$$\begin{aligned} \ln(1 + \eta'_n(\Lambda)) = & \frac{-4\text{Re}\sqrt{1 - (\Lambda + i n |u|)^2} - n(2\mu + 4u)}{T} \\ & + \int_{-\pi}^{\pi} dk \cos k a_n(\sin k - \Lambda) \ln \left(1 + \frac{1}{\zeta(k)} \right) + \sum_{m=1}^{\infty} A_{nm} * \ln \left(1 + \frac{1}{\eta'_m} \right) \Big|_{\Lambda}. \end{aligned} \quad (2.3.36)$$

Here we have defined

$$\begin{aligned} \zeta(k) &= \rho^h(k) / \rho^p(k), \\ \eta_n(\Lambda) &= \sigma_n^h(\Lambda) / \sigma_n^p(\Lambda), \\ \eta'_n(\Lambda) &= \sigma_n'^h(\Lambda) / \sigma_n'^p(\Lambda). \end{aligned} \quad (2.3.37)$$

Equations (2.3.34) to (2.3.36) are the TBA equations for the 1D attractive Hubbard model. Those equations are similar to the repulsive case, despite the difference of signs of $\ln(1 + \zeta^{-1}(k))$ terms in Equations (2.3.35) and (2.3.36) and that of the driving term in Equation (2.3.36).

Moreover, we introduce the dressed energies $\kappa(k)$, $\epsilon_n(\Lambda)$, and $\epsilon'_n(\Lambda)$

$$\kappa(k) = T \ln \zeta(k), \quad (2.3.38)$$

$$\epsilon_n(\Lambda) = T \ln \eta_n(\Lambda), \quad (2.3.39)$$

$$\epsilon'_n(\Lambda) = T \ln \eta'_n(\Lambda); \quad (2.3.40)$$

then the TBA equations are recast into

$$\begin{aligned} \kappa(k) = & -2 \cos k - \mu - 2u - B + T \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} d\Lambda a_n(\sin k - \Lambda) \ln \left(1 + e^{-\epsilon'_n(\Lambda)/T} \right) \\ & - T \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} d\Lambda a_n(\sin k - \Lambda) \ln \left(1 + e^{-\epsilon_n(\Lambda)/T} \right) \end{aligned} \quad (2.3.41)$$

$$\begin{aligned} \epsilon_n(\Lambda) = & 2nB + T \int_{-\pi}^{\pi} dk \cos k a_n(\sin k - \Lambda) \ln \left(1 + e^{-\kappa(k)/T} \right) \\ & + T \sum_{m=1}^{\infty} \int_{-\infty}^{\infty} \frac{dy}{2\pi} \frac{d}{d\Lambda} \Theta_{nm} \left(\frac{\Lambda - y}{|u|} \right) \ln \left(1 + e^{-\epsilon_m(y)/T} \right) \end{aligned} \quad (2.3.42)$$

$$\begin{aligned} \epsilon'_n(\Lambda) = & -4\text{Re}\sqrt{1 - (\Lambda + i n |u|)^2} - n(2\mu + 4u) + T \int_{-\pi}^{\pi} dk \cos k a_n(\sin k - \Lambda) \ln \left(1 + e^{-\kappa(k)/T} \right) \\ & + T \sum_{m=1}^{\infty} \int_{-\infty}^{\infty} \frac{dy}{2\pi} \frac{d}{d\Lambda} \Theta_{nm} \left(\frac{\Lambda - y}{|u|} \right) \ln \left(1 + e^{-\epsilon'_m(y)/T} \right). \end{aligned} \quad (2.3.43)$$

The Gibbs free energy per site eq. (2.3.22) could be expressed in terms of dressed energies,

$$f = u - T \int_{-\pi}^{\pi} \frac{dk}{2\pi} \ln \left(1 + e^{-\kappa(k)/T} \right) - T \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} \frac{d\Lambda}{\pi} \operatorname{Re} \left[\frac{1}{\sqrt{1 - (\Lambda + i n |u|)^2}} \right] \ln \left(1 + e^{-\epsilon'_n(\Lambda)/T} \right). \quad (2.3.44)$$

The TBA equations eqs. (2.3.41) to (2.3.43) indicate that the dressed energies $\kappa(k)$, $\epsilon_n(\Lambda)$, $\epsilon'_n(\Lambda)$ describe the excitation energies which are subject to interactions among the bound states of electrons, spin wave fluctuations, magnetic field and chemical potential. They contain full thermal and magnetic fluctuations in both spin and charge degrees of freedom. Therefore from these equations we can determine the thermal and magnetic properties of the model in the full temperature regimes.

We then transform the above TBA equations into another formulation with the help of the inverse of A_{nm} , which is defined as follow

$$A_{kn}^{-1} * f|_x = \delta_{k,n} f(x) - (\delta_{k-1,n} + \delta_{k+1,n}) \int_{-\infty}^{\infty} dy s(x-y) f(y). \quad (2.3.45)$$

Here the function $s(x)$ reads

$$s(x) = \frac{1}{4|u| \cosh(\pi x/2|u|)} = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \frac{\exp(-i\omega x)}{2 \cosh(\omega|u|)}. \quad (2.3.46)$$

Applying A_{nm}^{-1} on both sides of the m -th equation in eq. (2.3.35), and summing up these equations over m from 1 to infinity gives

$$\begin{aligned} & \sum_{m=1}^{\infty} A_{nm}^{-1} * \ln(1 + \eta_m) \Big|_{\Lambda} \\ &= \ln[1 + \eta_n^{-1}(\Lambda)] + \sum_{m=1}^{\infty} \int_{-\infty}^{\infty} dy A_{nm}^{-1}(\Lambda - y) \left\{ \frac{2mB}{T} + \int_{-\pi}^{\pi} dk \cos k a_m(\sin k - y) \ln[1 + \zeta^{-1}(k)] \right\}. \end{aligned} \quad (2.3.47)$$

Substituting eq. (2.3.45) into eq. (2.3.47) and after some manipulations, then gives

$$\begin{aligned} & \ln[1 + \eta_n^{-1}(\Lambda)] \\ &= \sum_{m=1}^{\infty} \left\{ \delta_{n,m} \ln[1 + \eta_m(\Lambda)] - (\delta_{n-1,m} + \delta_{n+1,m}) \int_{-\infty}^{\infty} dy s(\Lambda - y) \ln[1 + \eta_m(y)] \right\} \\ & \quad - \sum_{m=1}^{\infty} \int_{-\infty}^{\infty} dy [\delta_{n,m} \delta(y) - (\delta_{n-1,m} + \delta_{n+1,m}) s(y)] m \frac{2B}{T} \end{aligned}$$

$$- \sum_{m=1}^{\infty} \int_{-\pi}^{\pi} dk \cos k \ln[1 + \zeta^{-1}(k)] \int_{-\infty}^{\infty} dy A_{nm}^{-1}(\Lambda - y) a_m(\sin k - y). \quad (2.3.48)$$

If $n \geq 2$ in the above equation, going on with the calculation yields

$$\ln \eta_n(\Lambda) = s * \ln[(1 + \eta_{n-1})(1 + \eta_{n+1})] \Big|_{\Lambda}. \quad (2.3.49)$$

If $n = 1$, it results in

$$\ln \eta_1(\Lambda) = s * \ln(1 + \eta_2) \Big|_{\Lambda} + \int_{-\pi}^{\pi} dk \cos k s(\Lambda - \sin k) \ln[1 + \zeta^{-1}(k)]. \quad (2.3.50)$$

In the above derivation, we have used eq. (B.3.5) from Appendix B.

Through similar treatment of eq. (2.3.36),

$$\ln \eta'_n(\Lambda) = s * \ln[(1 + \eta'_{n-1})(1 + \eta'_{n+1})] \quad \text{if } n \geq 2. \quad (2.3.51)$$

$$\ln \eta'_1(\Lambda) = s * \ln(1 + \eta'_2) \Big|_{\Lambda} + \int_{-\pi}^{\pi} dk \cos k s(\Lambda - \sin k) \ln[1 + \zeta(k)]. \quad (2.3.52)$$

Now it is the turn of $\ln \zeta(k)$. From eqs. (2.3.35) and (2.3.36) we could derive that

$$\begin{aligned} & \ln \frac{1 + \eta'_n(\Lambda)}{1 + \eta_n(\Lambda)} \\ &= \frac{-4\text{Re}\sqrt{1 - (\Lambda + i n |u|)^2} - n(2\mu + 4u) - 2nB}{T} + \sum_{m=1}^{\infty} A_{nm} * \ln \left(\frac{1 + \eta_m'^{-1}}{1 + \eta_m^{-1}} \right) \Big|_{\Lambda}. \end{aligned} \quad (2.3.53)$$

Applying A^{-1} on both sides of eq. (2.3.53) and after some manipulations,

$$\begin{aligned} & \ln \frac{1 + \eta_m'^{-1}(\Lambda)}{1 + \eta_m^{-1}(\Lambda)} \\ &= \frac{1}{T} \sum_{m=1}^{\infty} \int_{-\infty}^{\infty} dy A_{nm}^{-1}(\Lambda - y) \left[4\text{Re}\sqrt{1 - (y + i m |u|)^2} + m(2\mu + 4u) + 2mB \right] \\ &+ \sum_{m=1}^{\infty} A_{nm}^{-1} * \ln \left(\frac{1 + \eta'_m}{1 + \eta_m} \right) \Big|_{\Lambda}, \end{aligned} \quad (2.3.54)$$

substituting which into eq. (2.3.34) yields

$$\ln \zeta(k) = - \frac{2 \cos k}{T} + \int_{-\infty}^{\infty} dy s(y - \sin k) \ln \frac{1 + \eta'_1(y)}{1 + \eta_1(y)}$$

$$+ \frac{1}{T} \int_{-\infty}^{\infty} dy 4 \operatorname{Re} \sqrt{1 - (y + i|u|)^2} s(y - \sin k). \quad (2.3.55)$$

Now we have obtained an equivalent form of the TBA equations, expressed by eqs. (2.3.49) to (2.3.52) and (2.3.55). With this new form of TBA equations, we introduce two equalities under the limit $n \rightarrow \infty$,

$$\lim_{n \rightarrow \infty} \frac{\ln \eta_n}{n} = \frac{2B}{T}, \quad (2.3.56)$$

$$\lim_{n \rightarrow \infty} \frac{\ln \eta'_n}{n} = -\frac{2\mu}{T}. \quad (2.3.57)$$

2.3.1 Zero Temperature Limit

In this part, we study the properties of dressed energies and root densities in the zero temperature limit. To begin with, we introduce and prove some useful results:

$$\epsilon_n(\Lambda) > 0, n = 1, 2, 3, \dots \quad (2.3.58)$$

$$\lim_{T \rightarrow 0} \ln \left(1 + \frac{1}{\eta_n(\Lambda)} \right) = 0, n = 1, 2, 3, \dots \quad (2.3.59)$$

$$\epsilon'_n(\Lambda) > 0, n = 2, 3, 4, \dots \quad (2.3.60)$$

$$\lim_{T \rightarrow 0} \ln \left(1 + \frac{1}{\eta'_n(\Lambda)} \right) = 0, n = 2, 3, 4, \dots \quad (2.3.61)$$

Proof. We only prove eqs. (2.3.58) and (2.3.59), while eqs. (2.3.60) and (2.3.61) are accessible through a similar procedure.

Obviously, if $n \geq 2$ then $\epsilon_n(\Lambda) = T \ln \eta_n(\Lambda) \geq 0$. This can be seen through by virtue of the consecutive equation eq. (2.3.49). Here the equality may be achievable in the zero temperature limit.

Taking account of the limit eq. (2.3.56), we know that $\forall \epsilon > 0, \exists N_c \in \mathbb{N}$, s.t. $n > N_c$, then $\left| \frac{2B}{T} - \frac{\ln \eta_n}{n} \right| < \epsilon$. This implies that $(2B - T\epsilon)n < \epsilon_n < (2B + T\epsilon)n$. Now one could see that even under zero temperature limit, if $n > N_c$ then $\epsilon_n > 0$. We turn back to the consecutive equation eq. (2.3.49) and write down

$$\begin{aligned} \epsilon_{N_c} &= T s * \ln \left[\left(1 + e^{\epsilon_{N_c+1}/T} \right) \left(1 + e^{\epsilon_{N_c-1}/T} \right) \right] \\ &\geq T s * \ln \left(1 + e^{\epsilon_{N_c+1}/T} \right) \\ &\sim s * \epsilon_{N_c+1} \\ &> 0. \end{aligned} \quad (2.3.62)$$

By consecutive operations similar to the above, one could derive $\epsilon_{N_c-1} > 0$, $\epsilon_{N_c-2} > 0$, \dots , $\epsilon_2 > 0$.

We then prove $\epsilon_1 > 0$. Due to eq. (2.3.55), one has

$$\epsilon_1(\Lambda) = T \cdot s * \ln(1 + \eta_2)|_\Lambda + T \int_{-\pi}^{\pi} dk \cos k s(\Lambda - \sin k) \ln \left(1 + \frac{1}{\zeta(k)} \right). \quad (2.3.63)$$

The first term in the rhs is positive. For the second term, we make the transformation

$$\int_{-\pi}^{\pi} dk \cos k s(\Lambda - \sin k) \ln \left(1 + \frac{1}{\zeta(k)} \right) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dk \cos k s(\Lambda - \sin k) \ln \frac{1 + \frac{1}{\zeta(k)}}{1 + \frac{1}{\zeta(-k+\pi)}}. \quad (2.3.64)$$

It is clear that if $\ln \frac{1 + \frac{1}{\zeta(k)}}{1 + \frac{1}{\zeta(-k+\pi)}} > 0$, then the second term in the rhs is positive. Considering that

$$\ln \zeta(k) - \ln \zeta(\pi - k) = -\frac{4 \cos k}{T} \leq 0, \quad k \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

one confirms

$$0 < \zeta(k) \leq \zeta(\pi - k).$$

Hereby we obtain

$$\ln \frac{1 + \frac{1}{\zeta(k)}}{1 + \frac{1}{\zeta(-k+\pi)}} > 0,$$

and thus $\epsilon_1(\Lambda) > 0$ is proved.

By now, we have proved $\epsilon_n(\Lambda) > 0$, $n = 1, 2, 3, \dots$.

The eq. (2.3.59) follows eq. (2.3.58) due to the fact that

$$\lim_{T \rightarrow 0} \ln \left(1 + \frac{1}{\eta_n(\Lambda)} \right) = \lim_{T \rightarrow 0} \ln \left(1 + e^{-\epsilon_n(\Lambda)/T} \right) = 0. \quad (2.3.65)$$

Dressed Energies

Taking the limit $T \rightarrow 0$ and applying eqs. (2.3.59) and (2.3.61) in eqs. (2.3.41) to (2.3.43), we obtain the TBA equations in the zero temperature limit. For simplicity, we introduce the notation

$$\mathfrak{F}(x) = \mathfrak{F}^+(x) + \mathfrak{F}^-(x) \quad (2.3.66)$$

with

$$\mathfrak{F}^+(x) = \begin{cases} \mathfrak{F}^+(x), & \text{if } \mathfrak{F}^+(x) > 0 \\ 0 & \end{cases} \quad \text{and} \quad \mathfrak{F}^-(x) = \begin{cases} 0 \\ \mathfrak{F}^-(x), & \text{if } \mathfrak{F}^-(x) \leq 0. \end{cases} \quad (2.3.67)$$

We then write down the TBA equations in the zero temperature limit

$$\kappa(k) = -2 \cos k - \mu - 2u - B - \int_{-\infty}^{\infty} d\Lambda a_1(\sin k - \Lambda) \mathfrak{E}_1'^-(\Lambda), \quad (2.3.68)$$

$$\mathfrak{E}_n(\Lambda) = 2nB - \int_{-\pi}^{\pi} dk \cos k a_n(\Lambda - \sin k) \kappa^-(k), \quad (2.3.69)$$

$$\begin{aligned} \mathfrak{E}_1'(\Lambda) = & -2\mu - 2 \int_{-\pi}^{\pi} dk \cos^2 k a_1(\sin k - \Lambda) - \int_{-\pi}^{\pi} dk \cos k a_1(\sin k - \Lambda) \kappa^-(k) \\ & - \int_{-\infty}^{\infty} d\Lambda' a_2(\Lambda - \Lambda') \mathfrak{E}_1'^-(\Lambda'), \end{aligned} \quad (2.3.70)$$

$$\begin{aligned} \mathfrak{E}_n'(\Lambda) = & -4 \operatorname{Re} \sqrt{1 - (\Lambda + i n |u|)^2} - n(2\mu + 4u) - \int_{-\pi}^{\pi} dk \cos k a_n(\Lambda - \sin k) \kappa^-(k) \\ & + \int_{-\infty}^{\infty} d\Lambda' [a_{n-1}(\Lambda - \Lambda') + a_{n+1}(\Lambda - \Lambda')] \mathfrak{E}_1'^-(\Lambda'). \end{aligned} \quad (2.3.71)$$

In fact, in the zero temperature limit the dressed energies $\kappa(k)$ and $\mathfrak{E}_1'(\Lambda)$ are symmetric in their own domains. Moreover, they monotonously increase in $[0, \pi]$ and $[0, \infty)$, respectively. See the proof in Appendix C. These properties imply that assuming $\kappa(\pm Q) = 0$ with $Q \geq 0$, we have

$$\kappa(k) \begin{cases} < 0, & \text{if } |k| < Q \\ > 0, & \text{if } |k| > Q. \end{cases} \quad (2.3.72)$$

Similarly, with respect to $\mathfrak{E}_1'(\Lambda)$, if $\mathfrak{E}_1'(\pm A) = 0$, we know

$$\mathfrak{E}_1'(\Lambda) \begin{cases} < 0, & \text{if } |\Lambda| < A \\ > 0, & \text{if } |\Lambda| > A. \end{cases} \quad (2.3.73)$$

Root Densities

We move on to the study of root densities in the zero temperature limit. With the help of eqs. (2.3.58) to (2.3.61), one derives

$$\lim_{T \rightarrow 0} \frac{\sigma_n^p(\Lambda)}{\sigma_n^h(\Lambda)} = \lim_{T \rightarrow 0} \exp \left(-\frac{\mathfrak{E}_n(\Lambda)}{T} \right) = 0, \quad n \geq 1 \quad (2.3.74)$$

$$\lim_{T \rightarrow 0} \frac{\sigma_n'^p(\Lambda)}{\sigma_n'^h(\Lambda)} = \lim_{T \rightarrow 0} \exp \left(-\frac{\mathfrak{E}_n'(\Lambda)}{T} \right) = 0, \quad n \geq 2 \quad (2.3.75)$$

$$\lim_{T \rightarrow 0} \frac{\sigma_1'^P(\Lambda)}{\sigma_1'^h(\Lambda)} = \lim_{T \rightarrow 0} \exp\left(-\frac{\epsilon_1'(\Lambda)}{T}\right) = 0, \text{ if } |\Lambda| > A \quad (2.3.76)$$

$$\lim_{T \rightarrow 0} \frac{\rho^P(k)}{\rho^h(k)} = \lim_{T \rightarrow 0} \exp\left(-\frac{\kappa(k)}{T}\right) = 0, \text{ if } |k| > Q, \quad (2.3.77)$$

which means that

$$\sigma_n^P(\Lambda) = 0, n \geq 1 \quad (2.3.78)$$

$$\sigma_n'^P(\Lambda) = 0, n \geq 2 \quad (2.3.79)$$

$$\sigma_1'^P(\Lambda) = 0, \text{ if } |\Lambda| > A \quad (2.3.80)$$

$$\rho^P(k) = 0, \text{ if } |k| > Q. \quad (2.3.81)$$

Here we have reasonably assumed that the root densities are smooth, bounded functions. Exploiting the above results, eqs. (2.3.15) to (2.3.17) can be rewritten as

$$\rho^P(k) = \theta_H(Q - |k|) \left[\frac{1}{2\pi} - \cos k \int_{-A}^A d\Lambda a_1(\Lambda - \sin k) \sigma_1'(\Lambda) \right], \quad (2.3.82)$$

$$\begin{aligned} \sigma_1'^P(\Lambda) = \theta_H(A - |\Lambda|) & \left[\frac{1}{\pi} \operatorname{Re} \frac{1}{\sqrt{1 - (\Lambda + i|u|)^2}} - \int_{-Q}^Q dk a_1(\sin k - \Lambda) \rho^P(k) \right. \\ & \left. - \int_{-A}^A d\Lambda' a_2(\Lambda - \Lambda') \sigma_1'^P(\Lambda') \right], \end{aligned} \quad (2.3.83)$$

$$\rho^h(k) = \theta_H(|k| - Q) \left[\frac{1}{2\pi} - \cos k \int_{-A}^A d\Lambda a_1(\sin k - \Lambda) \sigma_1'^P(\Lambda) \right], \quad (2.3.84)$$

$$\begin{aligned} \sigma_1'^h(\Lambda) = \theta_H(|\Lambda| - A) & \left[\frac{1}{\pi} \operatorname{Re} \frac{1}{\sqrt{1 - (\Lambda + i|u|)^2}} - \int_{-Q}^Q dk a_1(\sin k - \Lambda) \rho^P(k) \right. \\ & \left. - \int_{-A}^A d\Lambda' a_2(\Lambda - \Lambda') \sigma_1'^P(\Lambda') \right], \end{aligned} \quad (2.3.85)$$

$$\sigma_n^h(\Lambda) = \int_{-Q}^Q dk a_n(\Lambda - \sin k) \rho^P(k), \quad (2.3.86)$$

$$\sigma_n'^h(\Lambda) = \frac{1}{\pi} \operatorname{Re} \frac{1}{\sqrt{1 - (\Lambda + i n |u|)^2}} - A_{n1} * \sigma_1'^P \Big|_{\Lambda} - \int_{-Q}^Q dk a_n(\Lambda - \sin k) \rho^P(k), n \geq 2, \quad (2.3.87)$$

where $\theta_H(x)$ is the Heaviside step function.

One may notice that in the zero temperature limit, the functional forms of $\rho^P(k)$ and $\rho^h(k)$ are identical, so are that of $\sigma_1'^P(\Lambda)$ and $\sigma_1'^h(\Lambda)$. Hence, it is

convenient to define

$$\rho(k) = \rho^p(k) + \rho^h(k), \quad (2.3.88)$$

$$\sigma'_1(\Lambda) = \sigma'^p_1(\Lambda) + \sigma'^h_1(\Lambda), \quad (2.3.89)$$

and thus we arrive at

$$\rho(k) = \frac{1}{2\pi} - \cos k \int_{-A}^A d\Lambda a_1(\sin k - \Lambda) \sigma'_1(\Lambda), \quad (2.3.90)$$

$$\sigma'_1(\Lambda) = \frac{1}{\pi} \text{Re} \frac{1}{\sqrt{1 - (\Lambda + i|u|)^2}} - \int_{-Q}^Q dk a_1(\sin k - \Lambda) \rho(k) - \int_{-A}^A d\Lambda' a_2(\Lambda - \Lambda') \sigma'_1(\Lambda'). \quad (2.3.91)$$

2.3.2 Ground State Phase Diagram

Integrating eqs. (2.3.90) and (2.3.91) yields the total number of particles per site and the number of spin-down particles per site, respectively

$$\int_{-Q}^Q dk \rho(k) + 2 \int_{-A}^A d\Lambda \sigma'_1(\Lambda) = \frac{N}{L}, \quad (2.3.92)$$

$$\int_{-A}^A d\Lambda \sigma'_1(\Lambda) = \frac{M}{L} = \frac{N_\downarrow}{L}. \quad (2.3.93)$$

Hence, the particle density n and the magnetization per site m are expressed as

$$n = \frac{N}{L} = \int_{-Q}^Q dk \rho(k) + 2 \int_{-A}^A d\Lambda \sigma'_1(\Lambda), \quad (2.3.94)$$

$$m = \frac{\langle S_z \rangle}{L} = \frac{N - 2M}{2L} = \frac{1}{2} \int_{-Q}^Q dk \rho(k). \quad (2.3.95)$$

According to the properties of dressed energies, we recast the integral equations for the dressed energies in the zero temperature limit into

$$\kappa(k) = -2 \cos k - \mu - 2u - B - \int_{-A}^A d\Lambda a_1(\sin k - \Lambda) \epsilon'_1(\Lambda), \quad (2.3.96)$$

$$\begin{aligned} \epsilon'_1(\Lambda) = & -2\mu - 2 \int_{-\pi}^{\pi} dk \cos^2 k a_1(\sin k - \Lambda) - \int_{-Q}^Q dk \cos k a_1(\sin k - \Lambda) \kappa(k) \\ & - \int_{-A}^A d\Lambda' a_2(\Lambda - \Lambda') \epsilon'_1(\Lambda'). \end{aligned} \quad (2.3.97)$$

By varying the integration boundaries Q and A , the system possesses different fillings and quantum phases. On the basis of eqs. (2.3.94) and (2.3.95), it is easy to see that $Q = \pi$ implies $A = 0$ and corresponds to a half-filled band, i.e., $N = L$;

on the other hand, $Q = 0$ implies that $m = 0$, i.e., the system possesses a non-polarized phase. In total, we can distinguish five phases in the zero temperature limit as follows.

Phase I: $Q = 0, A = 0$. Vacuum. It is easy to see that $Q = 0$ and $A = 0$ imply an empty band and thus $n_c = m = 0$. In this phase, the zero temperature TBA equations (2.3.96) and (2.3.97) simplify to

$$\kappa(k) = -2 \cos k - \mu - 2u - B, \quad (2.3.98)$$

$$\epsilon'_1(\Lambda) = -2\mu - 2 \int_{-\pi}^{\pi} dk \cos^2 k a_1(\sin k - \Lambda). \quad (2.3.99)$$

Phase II: $0 < Q < \pi, A = 0$. Fully polarized, and partially filled phase. This phase corresponds to the case that particle densities lie between zero (empty band) and one (half-filled band), i.e. $0 < n_c < 1$; and no spin-down particle exists (completely polarized), i.e., $M = 0$. Obviously, in this phase, $m = n_c/2$. The TBA equations are simplified as

$$\kappa(k) = -2 \cos(k) - \mu - 2u - B, \quad (2.3.100)$$

$$\epsilon'_1(\Lambda) = -2\mu - 2 \int_{-\pi}^{\pi} dk \cos^2 k a_1(\sin k - \Lambda) - \int_{-Q}^Q dk \cos k a_1(\sin k - \Lambda) \kappa(k). \quad (2.3.101)$$

Phase V: $Q = 0, 0 < A \leq \infty$. Fully paired and partially filled phase. The particle density in this phase also lies between zero and one, i.e., $0 < n_c < 1$. This phase is solely filled with $k - \lambda$ strings of length 1, which means that every two particles of opposite spins form a bound pair and thus there is no excess fermion. It is easy to know that $m = 0$ and $N = 2M$. The TBA equations in this phase are given by

$$\kappa(k) = -2 \cos k - \mu - 2u - B - \int_{-A}^A d\Lambda a_1(\sin k - \Lambda) \epsilon'_1(\Lambda), \quad (2.3.102)$$

$$\epsilon'_1(\Lambda) = -2\mu - 2 \int_{-\pi}^{\pi} dk \cos^2 k a_1(\sin k - \Lambda) - \int_{-A}^A d\Lambda' a_2(\Lambda - \Lambda') \epsilon'_1(\Lambda'). \quad (2.3.103)$$

Phase IV: $0 < Q < \pi, 0 < A \leq \infty$. Partially filled and partially polarized phase. This is a FFLO-like phase, see chapter 5. The particle density satisfies $0 < n_c < 1$. This band is partly filled with $k - \lambda$ strings of length 1, which means bound pairs and excess unpaired fermions coexist. No simplification could be made to the TBA equations (2.3.96) and (2.3.97).

Phase III: $Q = \pi, A = 0$. Half-filling and fully polarized phase. This phase correspond to the case where $n_c = 1$ and $m = n_c/2$. In this phase, there is no

spin-down particle. The TBA equations could be recast into

$$\kappa(k) = -2\cos k - \mu - 2u - B, \quad (2.3.104)$$

$$\epsilon'_1(\Lambda) = -2\mu - 2 \int_{-\pi}^{\pi} dk \cos^2 k a_1(\Lambda - \sin k) - \int_{-\pi}^{\pi} dk \cos k a_1(\sin k - \Lambda) \kappa(k). \quad (2.3.105)$$

Now we are in a position to determine the boundaries among these five phases. A phase transition occurs when the dressed energies exactly satisfy $\kappa(0) = 0$, $\kappa(\pi) = 0$ or $\epsilon'_1(0) = 0$.

PB(I, V) is determined by $\epsilon'_1(0) = 0$. By virtue of the simplified TBA equations in eq. (2.3.99), one can simply obtain

$$\mu_{c1} = 2|u| - 2\sqrt{1+u^2}. \quad (2.3.106)$$

PB(I, II) is determined by $\kappa(0) = 0$. Employing eq. (2.3.98), it is easy to derive

$$\mu_{c2} = -B - 2u - 2. \quad (2.3.107)$$

PB(II, III) is determined by $\kappa(\pi) = 0$. In light of eq. (2.3.104), we obtain

$$\mu_{c3} = 2 - B - 2u. \quad (2.3.108)$$

PB(II, IV) is determined by $\epsilon'_1(0) = 0$. Taking account of eq. (2.3.101), we obtain a parametric expression as follows

$$\mu_{c4} = 2|u| - 2\sqrt{1+u^2} - \int_{-Q}^Q dk \cos k a_1(\sin k) [\cos Q - \cos k], \quad (2.3.109)$$

$$B_{c4} = 2\sqrt{1+u^2} - 2\cos Q - \int_{-Q}^Q dk \cos k a_1(\sin k) [\cos Q - \cos k], \quad (2.3.110)$$

with $Q \in [0, \pi]$.

PB(IV, V) is determined by $\kappa(0) = 0$, yet its calculation is subtle. We use the simplified TBA equations in Phase V, and the phase boundary can be calculated from the equations

$$\epsilon'_1(\Lambda) = -2\mu - 2 \int_{-\pi}^{\pi} dk \cos^2 k a_1(\sin k - \Lambda) - \int_{-A}^A d\Lambda' a_2(\Lambda - \Lambda') \epsilon'_1(\Lambda'), \quad (2.3.111)$$

$$\epsilon'_1(A) = 0, \quad (2.3.112)$$

$$\mu = -2 - 2u - B - \int_{-A}^A d\Lambda a_1(\Lambda) \epsilon'_1(\Lambda). \quad (2.3.113)$$

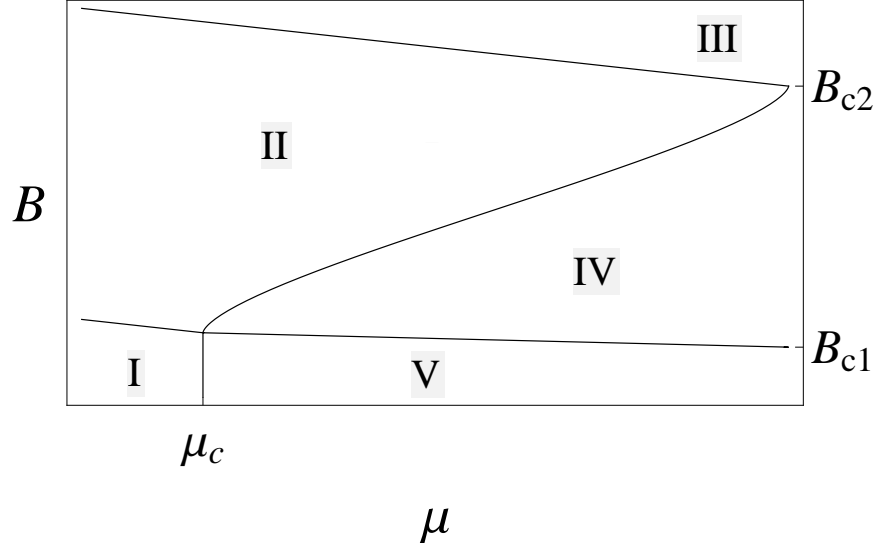


Figure 2.2: Ground state phase diagram of the 1D attractive Hubbard model in the μ - B plane. In the phase diagram, the critical fields are $\mu_c = 2|u| - 2\sqrt{1+u^2}$, $B_{c1} = 2|u| - 2 + 2 \int_{-\infty}^{\infty} d\omega \frac{J_1(\omega) \exp(-|u|\omega)}{w \cosh(u\omega)}$ and $B_{c2} = 2 + 2|u|$. Here $J_n(\omega)$ is the Bessel function of first kind. The five phase are denoted as, (I) vacuum, (II) fully polarized phase, (III) half-filling phase, (IV) partially polarized phase, and (V) fully paired phase.

When $A \ll 1$, the phase boundary could be obtained by iteration, i.e., by applying Taylor expansion to eq. (2.3.111) with respect to Λ , it can be approximately resolved by iteration. The solution of eq. (2.3.112) gives A in terms of μ and B , then we derive the phase boundary by substituting the above results for A into eq. (2.3.111) and eq. (2.3.113). By iteration, we finally obtain

$$\mu_{c5} \approx 2|u| - B - 2 + \frac{4\sqrt{2}}{\pi|u|\alpha_1} \left[\mu_{c5} + 2(\sqrt{1+u^2} - |u|) \right]^{\frac{3}{2}}, \quad (2.3.114)$$

in which $\alpha_1 = \int_{-\pi}^{\pi} dk \frac{2|u| \cos^2 k (u^2 - 3 \sin^2 k)}{\pi(\sin^2 k + u^2)^3}$ is a factor representing the lattice effect.

If $A \gg 1$, the phase boundary is given by eqs. (D.0.22) and (D.0.28) in Appendix D, where we have used the Wiener-Hopf method to solve the TBA integral equations.

2.4 Conclusion

In this chapter starting from the hamiltonian describing a host of fermions in crystal lattices, we demonstrated the derivation of the Lieb-Wu equations and the TBA equations for the 1D attractive Hubbard model. A thorough study of the root densities and the TBA equations in the zero temperature limit has been

made, on the basis of which we work out the ground state phase diagram and determine the analytical phase boundaries. The ground state phase diagram of the 1D attractive Hubbard model consists of five phases, of which we will pay special attention to the partially polarized phase IV. When $T = 0$, this phase is filled with unpaired fermions and charge bound states composed of two fermions which can form molecules in the strong coupling limit. Once $T > 0$, the charge bound states of multi-particles (four fermions, six fermions \dots) come into being. The TBA equations of those charge bound states shown in eq. (2.3.51) are the major difficulty in later discussion of equations of state.

In fact, the root pattern of the Lieb-Wu equations in the attractive case is similar to that of the repulsive case. Both of them contain real k 's, k - Λ strings and Λ - Λ strings. The difference occurs in the ground state where for the 1D attractive Hubbard model only real k 's and k - Λ strings of length one are permitted. While for the 1D repulsive Hubbard model, the k - Λ strings of length one are replaced by the Λ - Λ strings of length one. In addition, the phase diagram in the attractive case can be obtained by rotating 90 degrees from the repulsive case.

Low Temperature thermodynamics

The TBA equations describe the full thermodynamics of the 1D attractive Hubbard model. At low temperatures quantum liquid behavior and critical scaling in the thermodynamics should be obtained from the TBA equations eqs. (2.3.41) to (2.3.43). However, such coupled nonlinear integral equations provide a formidable challenge. In particular, it is challenging to solve infinitely many coupled nonlinear integral TBA equations either analytically or numerically. This obstacle prevents us to understand the microscopic Cooper pairing mechanism and many-body phenomena for this model. As far as we know, the Bosonization or Tomonaga-Luttinger liquid (TLL) theory is not available once the ferromagnetic ordering is involved in the low temperature physics. Neither of them is applicable to the quantum critical region near a phase transition.

In this chapter, we proceed with further investigation of the low energy physics of the 1D attractive Hubbard model beyond the scope of the Bosonization and TLL approaches [79–84]. In order to obtain the universal thermodynamics and quantum criticality of the 1D attractive Hubbard model, we first solve the TBA equations analytically in the strong coupling regime, and then derive the equation of state which is crucial for the investigation of the quantum criticality of the model. In the following discussion we mainly concentrate on the low density regime.

3.1 Equations of State

3.1.1 Preparation

Prior to later discussions, we denote the operator \hat{T} to simplify the TBA equations (2.3.41) to (2.3.43)

$$\hat{T}_{nm} * f|_x = \int_{-\infty}^{\infty} \frac{dy}{2\pi} \frac{d}{dx} \Theta_{nm} \left(\frac{x-y}{|u|} \right) f(y) \quad (3.1.1)$$

Recalling the definition of pressure of constant volume in thermodynamics, the pressure per unit length reads

$$p = - \left(\frac{\partial G}{\partial L} \right)_{T,N} = -f, \quad (3.1.2)$$

where f is expressed in eq. (2.3.44). We then introduce two new notations,

$$p^u = T \int_{-\pi}^{\pi} \frac{dk}{2\pi} \ln \left(1 + e^{-\kappa(k)/T} \right), \quad (3.1.3)$$

$$p_n^b = T \int_{-\infty}^{\infty} \frac{d\Lambda}{\pi} \operatorname{Re} \frac{1}{\sqrt{1 - (\Lambda + i n |u|)^2}} \ln \left(1 + e^{-\epsilon'_n(\Lambda)/T} \right), \quad (3.1.4)$$

and thus

$$p = p^u + \sum_{n=1}^{\infty} p_n^b + |u|, \quad (3.1.5)$$

where p^u and p_n^b represent the pressure arising from unpaired fermions and bound states of multi-particles, respectively. This formula provides an intuitively physical explanation for the origin of pressure.

From now on, we mainly focus on the strong correlated regime, i.e., $|u| \gg 1$ and retain terms up to $o\left(\frac{1}{|u|^4}\right)$.

3.1.2 Treatment of $\kappa(k)$

For convenience, we denote

$$\Delta_n(\Lambda) = \operatorname{Re} \frac{1}{\sqrt{1 - (\Lambda - i n |u|)^2}} = \frac{1}{2} \int_{-\pi}^{\pi} dk a_n(\Lambda - \sin k), \quad (3.1.6)$$

which can be expressed by the following series in the strong coupling regime

$$\Delta_n(\Lambda) = \pi a_n(\Lambda) \left[1 - \frac{1}{2} \frac{1}{(n|u|)^2 + \Lambda^2} \right] + o\left(\frac{1}{|u|^4}\right). \quad (3.1.7)$$

Starting from those integral terms involving $\epsilon'_n(\Lambda)$ on the rhs of eq. (2.3.41), we have

$$\begin{aligned} & \sum_{n=1}^{\infty} T \int_{-\infty}^{\infty} d\Lambda a_n(\sin k - \Lambda) \ln \left(1 + e^{-\epsilon'_n(\Lambda)/T} \right) \\ &= \sum_{n=1}^{\infty} T \int_{-\infty}^{\infty} d\Lambda \frac{a_n(\sin k - \Lambda)}{\Delta_n(\Lambda)} \Delta_n(\Lambda) \ln \left(1 + e^{-\epsilon'_n(\Lambda)/T} \right) \end{aligned}$$

$$\begin{aligned}
&= \sum_{n=1}^{\infty} T \int_{-\infty}^{\infty} d\Lambda \Delta_n(\Lambda) \ln \left(1 + e^{-\epsilon'_n(\Lambda)/T} \right) \frac{a_n(\Lambda) \left\{ 1 + \frac{2\Lambda \sin k - \sin^2 k}{(nu)^2 + \Lambda^2} + \left[\frac{2\Lambda \sin k - \sin^2 k}{(nu)^2 + \Lambda^2} \right]^2 + \left[\frac{2\Lambda \sin k - \sin^2 k}{(nu)^2 + \Lambda^2} \right]^3 \right\}}{\pi a_n(\Lambda) \left\{ 1 - \frac{1}{2} \frac{1}{(nu)^2 + \Lambda^2} + \frac{2\Lambda^2}{[(nu)^2 + \Lambda^2]^2} \right\}} \\
&= \sum_{n=1}^{\infty} T \int_{-\infty}^{\infty} \frac{d\Lambda}{\pi} \Delta_n(\Lambda) \ln \left(1 + e^{-\epsilon'_n(\Lambda)/T} \right) \left[1 + \frac{1}{2} \cos(2k) \frac{1}{(nu)^2 + \Lambda^2} \right] + o \left(\frac{1}{|u|^4} \right) \\
&= \sum_{n=1}^{\infty} p_n^b + \frac{1}{2} \cos(2k) T \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} \frac{d\Lambda}{\pi} \frac{\Delta_n(\Lambda)}{(nu)^2 + \Lambda^2} \ln \left(1 + e^{-\epsilon'_n(\Lambda)/T} \right) \\
&\quad - 2 \cos(2k) T \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} \frac{d\Lambda}{\pi} \frac{\Delta_n(\Lambda) \Lambda^2}{[(nu)^2 + \Lambda^2]^2} \ln \left(1 + e^{-\epsilon'_n(\Lambda)/T} \right) + o \left(\frac{1}{|u|^4} \right), \\
&= \sum_{n=1}^{\infty} p_n^b + \bar{a} + 2\bar{a} \cos^2 k + o \left(\frac{1}{|u|^4} \right), \tag{3.1.8}
\end{aligned}$$

where we have twice inserted the approximation of $\Delta_n(\Lambda)$, and

$$\bar{a} = \frac{1}{2} \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} d\Lambda \left[b_n(\Lambda) - \frac{4b_n(\Lambda)\Lambda^2}{(nu)^2 + \Lambda^2} \right] \ln \left(1 + e^{-\epsilon'_n(\Lambda)/T} \right), \tag{3.1.9}$$

with $b_n(\Lambda) = \frac{a_n(\Lambda)}{(nu)^2 + \Lambda^2}$.

With respect to the integrals regarding $\epsilon_n(\Lambda)$, it is easy to find that if $B/T \gg 1$ their summation is of exponential form and can be neglected. To make it clear, we rewrite the TBA equation for the dressed energy $\epsilon''(\Lambda)$ in eq. (2.3.42) as

$$\eta_n^{-1}(\Lambda) = \exp \left\{ -\frac{2nB}{T} - \int_{-\pi}^{\pi} dk \cos k a_n(\Lambda - \sin k) \ln \left(1 + e^{-\kappa(k)/T} \right) - \sum_{m=1}^{\infty} \hat{T}_{nm} * \ln \left(1 + \eta_n^{-1} \right) \Big|_{\Lambda} \right\}, \tag{3.1.10}$$

where the exponent is dominated by $-2nB/T$. This is obvious due to the fact that the convolution is definitely positive and the integral is of order $\frac{1}{|u|}$ in the strong coupling regime. Hence, one can make approximation

$$\eta_n^{-1}(\Lambda) \approx \exp \left\{ -\frac{2nB}{T} - 2\pi a_n(\Lambda) \bar{K} - \sum_{m=1}^{\infty} \hat{T}_{nm} * \ln \left(1 + \eta_n^{-1} \right) \Big|_{\Lambda} \right\}, \tag{3.1.11}$$

where $\bar{K} = \frac{1}{2\pi} \int_{-\pi}^{\pi} dk \cos k \ln \left(1 + e^{-\kappa(k)/T} \right)$. We turn back to the integrals in eq. (2.3.41),

$$\begin{aligned}
&T \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} d\Lambda a_n(\Lambda - \sin k) \ln \left(1 + \eta_n^{-1}(\Lambda) \right) \approx T \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} d\Lambda a_n(\Lambda) \ln \left(1 + \eta_n^{-1}(\Lambda) \right) \\
&\approx T \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} d\Lambda a_n(\Lambda) \eta_n^{-1}(\Lambda) \approx T \int_{-\infty}^{\infty} d\Lambda a_1(\Lambda) \eta_1^{-1}(\Lambda). \tag{3.1.12}
\end{aligned}$$

Substituting eq. (3.1.11) into which yields

$$\begin{aligned} T \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} d\Lambda a_1(\Lambda - \sin k) \ln(1 + \eta_n^{-1}(\Lambda)) &\approx T \int_{-\infty}^{\infty} d\Lambda a_1(\Lambda) \exp \left[-\frac{2B}{T} - 2\pi a_1(\Lambda) \bar{K} \right] \\ &\approx e^{-2B/T} T \int_{-\infty}^{\infty} d\Lambda a_1(\Lambda) \exp[-\bar{K} a_1(\Lambda)] \approx e^{-2B/T} T \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{d\theta}{\pi} \exp(-\bar{K} \cos^2 \theta), \end{aligned} \quad (3.1.13)$$

where we change the integral variable by letting $\frac{\Lambda}{n|u|} = \tan \theta$. Going on with the calculation, we arrive at

$$T \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} d\Lambda a_1(\Lambda - \sin k) \ln(1 + \eta_n^{-1}(\Lambda)) \approx T e^{-2B/T} e^{-\bar{K}} I_0(\bar{K}), \quad (3.1.14)$$

where $I_0(x)$ is the modified Bessel function of zero order.

After the above algebraic manipulations, now we can replace eq. (2.3.41) by

$$\kappa(k) = \kappa_0(k) - A^u, \quad (3.1.15)$$

where

$$\kappa_0(k) = -2 \cos k + 2\bar{a} \cos^2 k, \quad (3.1.16)$$

$$A^u = \mu + 2u + B - \sum_{n=1}^{\infty} p_n^b + \bar{a} + T e^{-2B/T} e^{-\bar{K}} I_0(\bar{K}). \quad (3.1.17)$$

Substitute eq. (3.1.15) into eq. (3.1.3), then integrate by parts, to obtain that

$$\begin{aligned} p^u &= \frac{T}{2\pi} \int_{-\pi}^{\pi} dk \ln(1 + e^{-\kappa(k)/T}) = \frac{T}{\pi} \int_0^{\pi} dk \ln(1 + e^{-\kappa(k)/T}) \\ &= \frac{T}{\pi} k \ln(1 + e^{-\kappa(k)/T}) \Big|_0^{\pi} + \frac{1}{\pi} \int_0^{\pi} dk \frac{1}{1 + \exp(\kappa(k)/T)} \frac{d\kappa(k)}{dk} \\ &= T \ln(1 + e^{-\kappa(\pi)/T}) + \frac{1}{\pi} \int_0^{\pi} dk \frac{1}{1 + \exp(\kappa(k)/T)} \frac{d\kappa_0(k)}{dk} \\ &= T \ln(1 + e^{-\kappa(\pi)/T}) + \frac{1}{\pi} \int_{\kappa_0(0)}^{\kappa_0(\pi)} d\kappa_0 \frac{k(\kappa_0)}{1 + \exp(\kappa_0/T)/z}, \end{aligned} \quad (3.1.18)$$

where $k(\kappa_0)$ is the inverse function of $\kappa_0(k)$, and

$$z = \exp(A^u/T), \quad (3.1.19)$$

$$\kappa(\pi) = 2 - \mu - 2u - B + \sum_{n=1}^{\infty} p_n^b + \bar{a}. \quad (3.1.20)$$

We would like to change the integral variable for eq. (3.1.18) by letting $\kappa_0 = 2x$, and go on with the calculation of p^μ ,

$$p^\mu = T \ln \left(1 + e^{-\kappa(\pi)/T} \right) + \frac{1}{\pi} \int_{-1+\bar{a}}^{1+\bar{a}} dx \frac{2 \cdot k(2x)}{1 + \exp(x/t)/z} \quad (3.1.21)$$

with

$$t = \frac{T}{2}. \quad (3.1.22)$$

Apparently, here we need an explicit expression for $k(2x)$. Through the defining equation $\kappa_0(k) = -2 \cos k + 2\bar{a} \cos^2 k = 2x$, one confirms that

$$\cos k = \frac{1 - \sqrt{1 + 4\bar{a}x}}{2\bar{a}}, \quad (3.1.23)$$

and hereby

$$k(2x) = \arccos \left(\frac{1 - \sqrt{1 + 4\bar{a}x}}{2\bar{a}} \right). \quad (3.1.24)$$

The other root

$$\cos k = \frac{1 + \sqrt{1 + 4\bar{a}x}}{2\bar{a}},$$

is dropped because it exceeds $[-1, 1]$ due to $-1 + \bar{a} < x < 1 + \bar{a}$ and $\bar{a} \ll 1$.

Here eqs. (3.1.21) and (3.1.24) are far from satisfactory, taking consideration of \bar{a} appearing in the limits of the integral shown in eq. (3.1.21) and in the argument of the \arccos function shown in eq. (3.1.24). To this end, we consider the Taylor expansion

$$\begin{aligned} \mathfrak{F}(\bar{a}) &= \int_{h(\bar{a})}^{g(\bar{a})} dy f(y, \bar{a}) \\ &= \int_{g(0)}^{h(0)} dy f(y, 0) + \bar{a} \cdot \mathfrak{F}'(\bar{a})|_{\bar{a}=0} + \frac{\bar{a}^2}{2!} \mathfrak{F}''(\bar{a})|_{\bar{a}=0} + \dots \end{aligned} \quad (3.1.25)$$

where

$$\mathfrak{F}'(\bar{a}) = \int_{g(\bar{a})}^{h(\bar{a})} dy \frac{\partial f(y, \bar{a})}{\partial \bar{a}} + f(h(\bar{a}), \bar{a}) \cdot \frac{dh(\bar{a})}{d\bar{a}} - f(g(\bar{a}), \bar{a}) \cdot \frac{dg(\bar{a})}{d\bar{a}} \quad (3.1.26)$$

and thus

$$\mathfrak{F}'(\bar{a})|_{\bar{a}=0} = \int_{g(0)}^{h(0)} dy \frac{\partial f(y, \bar{a})}{\partial \bar{a}} \Big|_{\bar{a}=0} + f(h(0), 0) \frac{dh(\bar{a})}{d\bar{a}} \Big|_{\bar{a}=0} - f(g(0), 0) \frac{dg(\bar{a})}{d\bar{a}} \Big|_{\bar{a}=0} \quad (3.1.27)$$

To apply this Taylor expansion, one has to reveal the reliance of $k(2x)$ upon \bar{a} , and hereby we denote

$$\bar{k}(x, \bar{a}) = k(2x). \quad (3.1.28)$$

Hence, the integral on the rhs of eq. (3.1.21) could be treated as a function of \bar{a} , and thus we expand it by using eq. (3.1.25), with result

$$\begin{aligned} & \frac{2}{\pi} \int_{-1+\bar{a}}^{1+\bar{a}} dx \frac{k(2x)}{1 + \exp(x/t)/z} = \frac{2}{\pi} \int_{-1+\bar{a}}^{1+\bar{a}} dx \frac{\bar{k}(x, \bar{a})}{1 + \exp(x/t)/z} \\ &= \frac{2}{\pi} \int_{-1}^1 dx \frac{\bar{k}(x, 0)}{1 + \exp(x/t)/z} + \frac{2}{\pi} \bar{a} \left\{ \int_{-1}^1 dx \frac{1}{1 + \exp(x/t)/z} \left(\frac{\partial \bar{k}(x, \bar{a})}{\partial \bar{a}} \Big|_{\bar{a}=0} \right) + \frac{\bar{k}(x, 0)}{1 + \exp(x/t)/z} \Big|_{x=1} \right. \\ & \quad \left. - \frac{\bar{k}(x, 0)}{1 + \exp(x/t)/z} \Big|_{x=-1} + \dots \right\} \\ &= \frac{2}{\pi} \int_{-1}^1 dx \frac{\arccos(-x)}{1 + \exp(x/t)/z} + \frac{2}{\pi} \bar{a} \left\{ \int_{-1}^1 dx \frac{1}{1 + \exp(x/t)/z} \left(\frac{\frac{x}{\bar{a}\sqrt{1+4\bar{a}x}} + \frac{1-\sqrt{1+4\bar{a}x}}{2\bar{a}^2}}{\sqrt{1 - \left(\frac{1-\sqrt{1+4\bar{a}x}}{2\bar{a}} \right)^2}} \Big|_{\bar{a}=0} \right) \right. \\ & \quad \left. + \frac{\pi}{1 + \exp(1/t)/z} - \frac{0}{1 + \exp(-1/t)/z} \right\} + \dots \\ &= \frac{2}{\pi} \int_{-1}^1 dx \frac{\arccos(-x)}{1 + \exp(x/t)/z} - \bar{a} \cdot \frac{2}{\pi} \int_{-1}^1 dx \frac{x^2/\sqrt{1-x^2}}{1 + \exp(x/t)/z} + \frac{2\bar{a}}{1 + \exp(1/t)/z} + \dots, \end{aligned} \quad (3.1.29)$$

where factually $\exp(1/t)/z = \exp(\kappa(\pi)/T)$.

As a consequence, the pressure p^u can be expressed as

$$\begin{aligned} p^u &= T \ln \left(1 + e^{-\kappa(\pi)/T} \right) + \frac{2}{\pi} \int_{-1}^1 dx \frac{\arccos(-x)}{1 + \exp(x/t)/z} - \bar{a} \cdot \frac{2}{\pi} \int_{-1}^1 dx \frac{x^2/\sqrt{1-x^2}}{1 + \exp(x/t)/z} \\ & \quad + \frac{2\bar{a}}{1 + \exp(\kappa(\pi)/t)} + o\left(\frac{1}{|u|^4}\right) \end{aligned} \quad (3.1.30)$$

where t , z , \bar{a} and $\kappa(\pi)$ have been defined by eq. (3.1.22), eq. (3.1.19), eq. (3.1.9) and eq. (3.1.20) beforehand. Obviously, the first two terms on the rhs of eq. (3.1.30) play the central role in pressure p^u .

3.1.3 Treatment of $\epsilon'_n(\Lambda)$

We move on to dealing with $\epsilon'_n(\Lambda)$. The technique is similar to what we have used in the last section for $\kappa(k)$. To begin with, we expand the driving terms of

eq. (2.3.43) in the strong coupling regime, and obtain

$$\begin{aligned}
\epsilon'_n(\Lambda) &= -4\text{Re}\sqrt{1 - (\Lambda + i n |u|)^2} - n(2\mu + 4u) + T \int_{-\pi}^{\pi} dk \cos k a_n(\sin k - \Lambda) \ln \left(1 + e^{-\kappa(k)/T}\right) \\
&\quad + \hat{T}_{nm} * \ln \left(1 + e^{-\epsilon'_n/T}\right) \Big|_{\Lambda} \\
&= -2n\mu - 2 \int_{-\pi}^{\pi} dk \cos^2 k a_n(\Lambda - \sin k) + T \int_{-\pi}^{\pi} dk \cos k a_n(\sin k - \Lambda) \ln \left(1 + e^{-\kappa(k)/T}\right) \\
&\quad + \hat{T}_{nm} * \ln \left(1 + e^{-\epsilon'_n/T}\right) \Big|_{\Lambda} \\
&= -2n\mu - 2 \int_{-\pi}^{\pi} dk \cos^2 k a_n(\Lambda) \left[1 - \frac{\sin^2 k}{(nu)^2 + \Lambda^2}\right] + a_n(\Lambda) T \int_{-\pi}^{\pi} dk \cos k \ln \left(1 + e^{-\kappa(k)/T}\right) \\
&\quad + b_n(\Lambda) T \int_{-\pi}^{\pi} dk \cos k \sin^2 k \ln \left(1 + e^{-\kappa(k)/T}\right) + \hat{T}_{nm} * \ln \left(1 + e^{-\epsilon'_n/T}\right) \Big|_{\Lambda} + o\left(\frac{1}{|u|^4}\right) \\
&= -2n\mu - d_1 \cdot a_n(\Lambda) - d_2 \cdot b_n(\Lambda) + \hat{T}_{nm} * \ln \left(1 + e^{-\epsilon'_n/T}\right) \Big|_{\Lambda} + o\left(\frac{1}{|u|^4}\right), \tag{3.1.31}
\end{aligned}$$

where the second equality arises from the application of eq. (B.3.12) in Appendix B, and we have denoted that

$$d_1 = 2\pi - \int_{-\pi}^{\pi} dk \cos k \ln \left(1 + e^{-\kappa(k)/T}\right), \tag{3.1.32}$$

$$d_2 = -\frac{\pi}{2} - \int_{-\pi}^{\pi} dk \cos k \sin^2 k \ln \left(1 + e^{-\kappa(k)/T}\right). \tag{3.1.33}$$

With respect to the summation of convolution terms, the low density condition means that the cut-off of the dressed energy $\epsilon'_n(\Lambda)$ is small, therefore we can make the following approximation for the general convolution term,

$$\begin{aligned}
&\int_{-\infty}^{\infty} d\Lambda' a_p(\Lambda - \Lambda') \ln \left(1 + e^{-\epsilon'_q(\Lambda')/T}\right) \\
&= \int_{-\infty}^{\infty} d\Lambda' a_p(\Lambda') \ln \left(1 + e^{-\epsilon'_q(\Lambda')/T}\right) - \Lambda^2 \int_{-\infty}^{\infty} d\Lambda' b_p(\Lambda') \ln \left(1 + e^{-\epsilon'_q(\Lambda')/T}\right) + o\left(\frac{1}{|u|^4}\right). \tag{3.1.34}
\end{aligned}$$

Substituting this result into the summation of convolution terms in eq. (3.1.31) yields

$$\epsilon'_n(\Lambda) = -2n\mu - d_1 \cdot a_n(\Lambda) - d_2 \cdot b_n(\Lambda) + \sum_{n=1} \eta_n + \sum_{n=1} \Lambda^2 \varphi_n + o\left(\frac{1}{|u|^4}\right), \tag{3.1.35}$$

with

$$\eta_n = T \sum_{m=1}^{\infty} \int_{-\infty}^{\infty} d\Lambda \hat{f}_{nm}(\Lambda) \ln \left(1 + e^{-\epsilon'_m(\Lambda)/T} \right), \quad (3.1.36)$$

$$\phi_n = T \sum_{m=1}^{\infty} \int_{-\infty}^{\infty} d\Lambda \hat{Q}_{nm}(\Lambda) \ln \left(1 + e^{-\epsilon'_m(\Lambda)/T} \right), \quad (3.1.37)$$

where for the sake of convenient representation, we denote

$$\hat{Q}_{nm}(x) = \begin{cases} b_{|n-m|}(x) + 2b_{|n-m|+2}(x) + \cdots + 2b_{n+m-2}(x) + b_{n+m}(x) & \text{if } n \neq m \\ 2b_2(x) + 2b_4(x) + \cdots + 2b_{2n-2}(x) + b_{2n}(x) & \text{if } n = m. \end{cases} \quad (3.1.38)$$

Furthermore, we expand the $a_n(\Lambda)$ and $b_n(\Lambda)$ functions, and obtain

$$\epsilon'_n(\Lambda) = D_n \left(\frac{\Lambda}{n|u|} \right)^2 - A_n^b, \quad (3.1.39)$$

where

$$A_n^b = 2n\mu - \eta_n + \frac{d_1}{\pi n|u|} + \frac{d_2}{2\pi(n|u|)^3}, \quad (3.1.40)$$

$$D_n = \frac{d_1}{\pi n|u|} - (nu)^2 \phi_n. \quad (3.1.41)$$

We now insert eq. (3.1.39) into the definition of p_n^b and take integration by part

$$p_n^b = T \left[1 - \frac{1}{4(nu)^2} \right] \ln[1 + \exp(\frac{2n\mu}{T})] + \frac{2D_n}{\pi} \left[1 - \frac{1}{4(nu)^2} \right] \int_0^{\infty} dx \frac{\arctan(\sqrt{x})}{1 + e^{x/\tau_n}/\zeta_n} + o\left(\frac{1}{|u|^4}\right), \quad (3.1.42)$$

where

$$\tau_n = \frac{T}{D_n}, \quad (3.1.43)$$

$$\zeta_n = \exp \left[\frac{1}{T} \left(2n\mu - \eta_n + \frac{d_1}{\pi n|u|} + \frac{d_2}{2\pi(n|u|)^3} \right) \right]. \quad (3.1.44)$$

We have thus derived the expression for p_n^b as well as p^u , yet there exist some unknown variables including d_1 , d_2 , \bar{a} , η_n , and ϕ_n . We find that the integrals present in these variables are of the form similar to p^u or p_n^b , and hence we can

deal with them by the same technique, with result

$$\begin{aligned} d_1 &= 2\pi - 4 \int_{-1}^1 dx \frac{\sqrt{1-x^2}}{1 + \exp(x/t)/z} - 4\bar{a} \int_{-1}^1 dx \frac{x^3/\sqrt{1-x^2}}{1 + \exp(x/t)/z} + o\left(\frac{1}{u^4}\right), \\ d_2 &= -\frac{\pi}{2} - \frac{4}{3} \int_{-1}^1 dx \frac{(1-x^2)^{3/2}}{1 + \exp(x/t)/z} + o\left(\frac{1}{u^4}\right), \end{aligned} \quad (3.1.45)$$

and

$$\bar{a} = \sum_{n=1}^{\infty} \frac{D_n}{\pi(nu)^2} \int_0^{\infty} dx \frac{\sqrt{x}/(1+x)^2}{1 + e^{D_n x/T}/\zeta_n} + o\left(\frac{1}{u^4}\right), \quad (3.1.46)$$

$$\begin{aligned} \eta_n &= \sum_{m=1}^{\infty} \mathfrak{T}_{nm}(\xi^m) + o\left(\frac{1}{u^4}\right) + o\left(\frac{1}{u^4}\right), \\ \phi_n &= \sum_{m=1}^{\infty} \mathfrak{T}_{nm}(\phi^m) + o\left(\frac{1}{u^6}\right) + o\left(\frac{1}{u^6}\right). \end{aligned} \quad (3.1.47)$$

where we have defined a new function $\mathfrak{T}_{nm}(x^m) = x_{|n-m|}^m + 2x_{|n-m|+2}^m + \cdots + 2x_{n+m-2}^m + x_{n+m}^m$ with $x_0^m = 0$, and auxiliary functions

$$\begin{aligned} \xi_p^m &= T \ln \left(1 + e^{2m\mu/T}\right) + \frac{2D_m}{\pi} \int_0^{\infty} dx \frac{\arctan\left(\frac{m}{p}\sqrt{x}\right)}{1 + e^{D_m x/T}/\zeta_m}, \\ \phi_p^m &= \frac{T}{2(pu)^2} \left(1 + e^{2m\mu/T}\right) + \frac{m}{p} \frac{D_m}{\pi u^2} \int_0^{\infty} dx \frac{\sqrt{x}/(p^2 + m^2 x)}{1 + e^{D_m x/T}/\zeta_m} + \frac{D_m}{\pi(pu)^2} \int_0^{\infty} dx \frac{\arctan\left(\frac{m}{p}\sqrt{x}\right)}{1 + e^{D_m x/T}/\zeta_m}. \end{aligned} \quad (3.1.48)$$

3.1.4 Summary for Equations of State

It is apparent that eqs. (3.1.30) and (3.1.42) play the role of equations of state, with auxiliary (3.1.45) to (3.1.48). They are indicative of the sophisticated many-body effects induced by k - Λ strings of different lengths.

In order to conceive the universal behavior of the system, we need to further simplify the equations of state given in eqs. (3.1.30) and (3.1.42). To this end, we utilize the condition $|\frac{\mu}{T}| \gg 1$ and strong interaction $|u| \gg 1$, which suppress the large length k - Λ strings in this physical regime. We observe that no larger length- n k - Λ bound states than $n = 1$ exist in the partially polarized phase IV at low temperatures. Thereby the equation of state simplifies to $p = p^u + p^b + |u|$, where p^u and p^b are given by

$$p^u = T \ln \left(1 + e^{-\kappa(\pi)/T}\right) + \frac{2}{\pi} \int_{-1}^1 dx \frac{\arccos(-x)}{1 + e^{2x/T}/z} + o\left(\frac{1}{u^3}\right), \quad (3.1.49)$$

$$p^b = \frac{2D_1}{\pi} \int_0^\infty dx \frac{\arctan \sqrt{x}}{1 + e^{D_1 x/T}/\zeta} + o\left(\frac{1}{u^3}\right), \quad (3.1.50)$$

where $z = e^{(2-\kappa(\pi))/T}$, $\zeta = e^{(2\mu-\eta+\frac{d_1}{\pi|u|})/T}$ and the above auxiliary functions with $n = 1$ read

$$\varepsilon''(\pi) = 2 - (\mu + 2u + B - p^b), \quad (3.1.51)$$

$$D_1 = \frac{d_1}{\pi|u|} - u^2 \varphi, \quad (3.1.52)$$

$$d_1 = 2\pi - 4 \int_{-1}^1 dx \frac{\sqrt{1-x^2}}{1 + e^{2x/T}/z} + o\left(\frac{1}{u^3}\right), \quad (3.1.53)$$

$$\eta = \frac{2D_1}{\pi} \int_0^\infty dx \frac{\arctan(\frac{1}{2}\sqrt{x})}{1 + e^{D_1 x/T}/\zeta} + o\left(\frac{1}{u^3}\right), \quad (3.1.54)$$

$$\varphi = \frac{D_1}{2\pi u^2} \int_0^\infty dx \frac{\sqrt{x}/(4+x)}{1 + e^{D_1 x/T}/\zeta} + \frac{D_1}{4\pi u^2} \int_0^\infty dx \frac{\arctan(\frac{1}{2}\sqrt{x})}{1 + e^{D_1 x/T}/\zeta} + o\left(\frac{1}{u^4}\right). \quad (3.1.55)$$

Hereafter we mainly consider the corrections up to order $1/|u|$ in the strong coupling regime $|u| \gg 1$. The equations of state (3.1.30) and (3.1.42) can give a very good approximation of the low energy physics.

3.2 Thermodynamic Quantities

The equations of state together with the standard thermodynamic relations afford access to the general expressions for thermodynamic quantities, which can be used to make comparison with experimental data and to investigate quantum criticality. For this purpose, we have to calculate the thermodynamic quantities which contain enough thermal and quantum fluctuations, and thus we use the form of the equations of state given in section 3.1.4. We observe that the first-order derivatives of these pressures with respect to μ or B form a set of linear equations. Solution to this set of linear equations directly leads to the particle density $n = \frac{\partial p}{\partial \mu}$ and magnetization $m = \frac{1}{2} \frac{\partial p}{\partial B}$. Similarly, one can derive the second-order derivatives of the pressures such as compressibility $\kappa_c = \frac{\partial^2 p}{\partial \mu^2}$ and susceptibility $\chi_s = \frac{1}{2} \frac{\partial^2 p}{\partial B^2}$.

To begin with, we rewrite eq. (3.1.50) approximately as

$$\begin{aligned} p^b &= \frac{2D_1}{\pi} \int_0^\infty dx \frac{\arctan(\sqrt{x})}{1 + \exp(x/\tau)/\zeta} \\ &\approx \frac{2D_1}{\pi} \int_0^\infty dx \frac{1}{1 + \exp(x/\tau)/\zeta} \left[x^{1/2} - \frac{1}{3} x^{3/2} + \frac{1}{5} x^{5/2} \right] \end{aligned}$$

$$= -\frac{D_1^{-\frac{1}{2}}}{\sqrt{\pi}} T^{\frac{3}{2}} \text{Li}_{3/2}(-\zeta) + \frac{D_1^{-\frac{3}{2}}}{2\sqrt{\pi}} T^{\frac{5}{2}} \text{Li}_{5/2}(-\zeta) - \frac{3D_1^{-\frac{5}{2}}}{4\sqrt{\pi}} T^{\frac{7}{2}} \text{Li}_{7/2}(-\zeta), \quad (3.2.1)$$

and similarly we can recast eqs. (3.1.54) and (3.1.55) as

$$\begin{aligned} \eta &= -\frac{D_1^{-\frac{1}{2}}}{2\sqrt{\pi}} T^{\frac{3}{2}} \text{Li}_{3/2}(-\zeta) + \frac{D_1^{-\frac{3}{2}}}{16\sqrt{\pi}} T^{\frac{5}{2}} \text{Li}_{5/2}(-\zeta) - \frac{3D_1^{-\frac{5}{2}}}{128\sqrt{\pi}} T^{\frac{7}{2}} \text{Li}_{7/2}(-\zeta), \quad (3.2.2) \\ u^2\varphi &= -\frac{D_1^{-\frac{1}{2}}}{8\sqrt{\pi}} T^{\frac{3}{2}} \text{Li}_{3/2}(-\zeta) + \frac{D_1^{-\frac{3}{2}}}{32\sqrt{\pi}} T^{\frac{5}{2}} \text{Li}_{5/2}(-\zeta) - \frac{9D_1^{-\frac{5}{2}}}{512\sqrt{\pi}} T^{\frac{7}{2}} \text{Li}_{7/2}(-\zeta). \end{aligned} \quad (3.2.3)$$

Hereafter, we denote $f_s = \text{Li}_s(-\zeta)$ with the polylogarithm defined as $\text{Li}_s(x) = \sum_{k=0}^{\infty} x^k / k^s$.

3.2.1 Derivatives with respect to μ

We start from differentiating p^b , η , $u^2\varphi$ with respect to μ ,

$$\begin{aligned} \frac{\partial p^b}{\partial \mu} &= \left(\frac{1}{2\sqrt{\pi}} \tau^{3/2} f_{3/2} - \frac{3}{4\sqrt{\pi}} \tau^{5/2} f_{5/2} + \frac{15}{8\sqrt{\pi}} \tau^{7/2} f_{7/2} \right) \frac{\partial D_1}{\partial \mu} \\ &\quad - \left(\frac{1}{\sqrt{\pi}} \tau^{1/2} f_{1/2} - \frac{1}{2\sqrt{\pi}} \tau^{3/2} f_{3/2} + \frac{3}{4\sqrt{\pi}} \tau^{5/2} f_{5/2} \right) \frac{T}{\zeta} \frac{\partial \zeta}{\partial \mu}, \end{aligned} \quad (3.2.4)$$

$$\begin{aligned} \frac{\partial \eta}{\partial \mu} &= \left(\frac{1}{4\sqrt{\pi}} \tau^{3/2} f_{3/2} - \frac{3}{32\sqrt{\pi}} \tau^{5/2} f_{5/2} + \frac{15}{256\sqrt{\pi}} \tau^{7/2} f_{7/2} \right) \frac{\partial D_1}{\partial \mu} \\ &\quad - \left(\frac{1}{2\sqrt{\pi}} \tau^{1/2} f_{1/2} - \frac{1}{16\sqrt{\pi}} \tau^{3/2} f_{3/2} + \frac{3}{128\sqrt{\pi}} \tau^{5/2} f_{5/2} \right) \frac{T}{\zeta} \frac{\partial \zeta}{\partial \mu}, \end{aligned} \quad (3.2.5)$$

$$\begin{aligned} \frac{\partial (u^2\varphi)}{\partial \mu} &= \left(\frac{1}{16\sqrt{\pi}} \tau^{3/2} f_{3/2} - \frac{3}{64\sqrt{\pi}} \tau^{5/2} f_{5/2} + \frac{45}{1024\sqrt{\pi}} \tau^{7/2} f_{7/2} \right) \frac{\partial D_1}{\partial \mu} \\ &\quad - \left(\frac{1}{8\sqrt{\pi}} \tau^{1/2} f_{1/2} - \frac{1}{32\sqrt{\pi}} \tau^{3/2} f_{3/2} + \frac{9}{512\sqrt{\pi}} \tau^{5/2} f_{5/2} \right) \frac{T}{\zeta} \frac{\partial \zeta}{\partial \mu}. \end{aligned} \quad (3.2.6)$$

In view of

$$\frac{\partial D_1}{\partial \mu} = \frac{1}{|u|} \frac{\partial (d_1/\pi)}{\partial \mu} - \frac{\partial (u^2\varphi)}{\partial \mu}, \quad (3.2.7)$$

$$\frac{T}{\zeta} \frac{\partial \zeta}{\partial \mu} = 2 - \frac{\partial \eta}{\partial \mu} + \frac{1}{|u|} \frac{\partial (d_1/\pi)}{\partial \mu}, \quad (3.2.8)$$

we substitute these equations into eqs. (3.2.4) to (3.2.6), after some algebraic

manipulations, and then obtain

$$\begin{aligned}
& \frac{\partial p^b}{\partial \mu} - \frac{\partial \eta}{\partial \mu} \left(\frac{1}{\sqrt{\pi}} \tau^{1/2} f_{1/2} - \frac{1}{2\sqrt{\pi}} \tau^{3/2} f_{3/2} + \frac{3}{4\sqrt{\pi}} \tau^{5/2} f_{5/2} \right) \\
& + \frac{\partial(u^2 \varphi)}{\partial \mu} \left(\frac{1}{2\sqrt{\pi}} \tau^{3/2} f_{3/2} - \frac{3}{4\sqrt{\pi}} \tau^{5/2} f_{5/2} + \frac{15}{8\sqrt{\pi}} \tau^{7/2} f_{7/2} \right) \\
& + \frac{\partial(d_1/\pi)}{\partial \mu} \frac{1}{|u|} \left(\frac{1}{\sqrt{\pi}} \tau^{1/2} f_{1/2} - \frac{1}{\sqrt{\pi}} \tau^{3/2} f_{3/2} + \frac{3}{2\sqrt{\pi}} \tau^{5/2} f_{5/2} - \frac{15}{8\sqrt{\pi}} \tau^{7/2} f_{7/2} \right) \\
& = -2 \left(\frac{1}{\sqrt{\pi}} \tau^{1/2} f_{1/2} - \frac{1}{2\sqrt{\pi}} \tau^{3/2} f_{3/2} + \frac{3}{4\sqrt{\pi}} \tau^{5/2} f_{5/2} \right), \tag{3.2.9}
\end{aligned}$$

$$\begin{aligned}
& \frac{\partial \eta}{\partial \mu} \left(1 - \frac{1}{2\sqrt{\pi}} \tau^{1/2} f_{1/2} + \frac{1}{16\sqrt{\pi}} \tau^{3/2} f_{3/2} - \frac{3}{128\sqrt{\pi}} \tau^{5/2} f_{5/2} \right) \\
& + \frac{\partial(u^2 \varphi)}{\partial \mu} \left(\frac{1}{4\sqrt{\pi}} \tau^{3/2} f_{3/2} - \frac{3}{32\sqrt{\pi}} \tau^{5/2} f_{5/2} + \frac{15}{256\sqrt{\pi}} \tau^{7/2} f_{7/2} \right) \\
& + \frac{\partial(d_1/\pi)}{\partial \mu} \frac{1}{|u|} \left(\frac{1}{2\sqrt{\pi}} \tau^{1/2} f_{1/2} - \frac{5}{16\sqrt{\pi}} \tau^{3/2} f_{3/2} + \frac{5}{128\sqrt{\pi}} \tau^{5/2} f_{5/2} - \frac{15}{256\sqrt{\pi}} \tau^{7/2} f_{7/2} \right) \\
& = -2 \left(\frac{1}{2\sqrt{\pi}} \tau^{1/2} f_{1/2} - \frac{1}{16\sqrt{\pi}} \tau^{3/2} f_{3/2} + \frac{3}{128\sqrt{\pi}} \tau^{5/2} f_{5/2} \right), \tag{3.2.10}
\end{aligned}$$

$$\begin{aligned}
& - \frac{\partial \eta}{\partial \mu} \left(\frac{1}{8\sqrt{\pi}} \tau^{1/2} f_{1/2} - \frac{1}{32\sqrt{\pi}} \tau^{3/2} f_{3/2} + \frac{9}{512\sqrt{\pi}} \tau^{5/2} f_{5/2} \right) \\
& + \frac{\partial(u^2 \varphi)}{\partial \mu} \left(1 + \frac{1}{16\sqrt{\pi}} \tau^{3/2} f_{3/2} - \frac{3}{64\sqrt{\pi}} \tau^{5/2} f_{5/2} + \frac{45}{1024\sqrt{\pi}} \tau^{7/2} f_{7/2} \right) \\
& + \frac{\partial(d_1/\pi)}{\partial \mu} \frac{1}{|u|} \left(\frac{1}{8\sqrt{\pi}} \tau^{1/2} f_{1/2} - \frac{3}{32\sqrt{\pi}} \tau^{3/2} f_{3/2} + \frac{33}{512\sqrt{\pi}} \tau^{5/2} f_{5/2} - \frac{45}{1024\sqrt{\pi}} \tau^{7/2} f_{7/2} \right) \\
& = -2 \left(\frac{1}{8\sqrt{\pi}} \tau^{1/2} f_{1/2} - \frac{3}{32\sqrt{\pi}} \tau^{3/2} f_{3/2} + \frac{9}{512\sqrt{\pi}} \tau^{5/2} f_{5/2} \right). \tag{3.2.11}
\end{aligned}$$

We turn to the derivative of d_1/π with respect to μ , and carry out the calculation

$$\frac{\partial(d_1/\pi)}{\partial \mu} = 2 \frac{T}{z} \frac{\partial z}{\partial \mu} \int_{-1}^1 dx \frac{x/\sqrt{1-x^2}}{1 + \exp(x/t)/z}. \tag{3.2.12}$$

In light of

$$\frac{T}{z} \frac{\partial z}{\partial \mu} = 1 - \frac{\partial p^b}{\partial \mu}, \tag{3.2.13}$$

eq. (3.2.12) can be rewritten as

$$2\delta \frac{\partial p^b}{\partial \mu} + \frac{\partial(d_1/\pi)}{\partial \mu} = 2\delta, \quad (3.2.14)$$

where we have denoted

$$\delta = \frac{1}{\pi} \int_{-1}^1 dx \frac{x/\sqrt{1-x^2}}{1 + \exp(x/t)/z}. \quad (3.2.15)$$

It's easy to see that, for eqs. (3.2.9) to (3.2.11) and (3.2.14), the derivatives of p^b , η , $u^2\varphi$, and d_1/π with respect to μ make up a set of linear equations. The solution of this set of equations is listed as follows,

$$\frac{\partial p^b}{\partial \mu} = - \frac{8\tau^{1/2} (2f_{1/2} - \tau f_{3/2})}{\Delta_t}, \quad (3.2.16)$$

$$\frac{\partial \eta}{\partial \mu} = - \frac{\tau^{1/2} (8f_{1/2} - \tau f_{3/2})}{\Delta_t}, \quad (3.2.17)$$

$$\frac{\partial(u^2\varphi)}{\partial \mu} = - \frac{\tau^{1/2} (4f_{1/2} - 3\tau f_{3/2})}{2\Delta_t}, \quad (3.2.18)$$

$$\frac{\partial(d_1/\pi)}{\partial \mu} = \frac{2\delta (8\sqrt{\pi} + 12\tau^{1/2} f_{1/2} - 7l\tau^{3/2} f_{3/2})}{\Delta_t}, \quad (3.2.19)$$

where we have omitted all $o\left(\frac{1}{|u|}\right)$ and $o(\tau^2)$ terms, and

$$\Delta_t = 8\sqrt{\pi} - 4\tau^{1/2} f_{1/2} + \tau^{3/2} f_{3/2}. \quad (3.2.20)$$

Similar to what we have done to d_1 above, the derivative of p^u with respect to μ reads

$$\frac{\partial p^u}{\partial \mu} = \left(1 - \frac{\partial p^b}{\partial \mu}\right) \gamma, \quad (3.2.21)$$

where we have used eqs. (3.1.51) and (3.2.13), and the fact that $\exp(\kappa(\pi)) = \exp(1/t)/z$. For simplicity, we have denoted

$$\gamma = \frac{1}{\pi} \int_{-1}^1 dx \frac{1}{1 + \exp(x/t)/z} \frac{1}{\sqrt{1-x^2}}. \quad (3.2.22)$$

The next stage is to calculate the second order derivative of p^b with respect

to μ . Prior to later calculations, we notice that

$$\frac{\partial}{\partial \mathbf{v}} (\tau^s f_s) = \frac{1}{D_1} \left[-s \tau^s f_s \frac{\partial D_1}{\partial \mathbf{v}} + \tau^{s-1} f_{s-1} \left(\frac{T}{\zeta} \frac{\partial \zeta}{\partial \mathbf{v}} \right) \right], \quad (3.2.23)$$

where \mathbf{v} could be either μ or B and we have used the polylogarithm property

$$\frac{\partial}{\partial z} \text{Li}_s(z) = \frac{1}{z} \text{Li}_{s-1}(z). \quad (3.2.24)$$

Differentiation of eqs. (3.2.9) to (3.2.11) and (3.2.14), and some algebraic manipulations, then leads to a set of linear equations,

$$\begin{aligned} & \frac{\partial^2 p^b}{\partial \mu^2} - \frac{\partial^2 \eta}{\partial \mu^2} \left(\frac{1}{\sqrt{\pi}} \tau^{1/2} f_{1/2} - \frac{1}{2\sqrt{\pi}} \tau^{3/2} f_{3/2} + \frac{3}{4\sqrt{\pi}} \tau^{5/2} f_{5/2} \right) \\ & + \frac{\partial^2 (u^2 \varphi)}{\partial \mu^2} \left(\frac{1}{2\sqrt{\pi}} \tau^{3/2} f_{3/2} - \frac{3}{4\sqrt{\pi}} \tau^{5/2} f_{5/2} + \frac{15}{8\sqrt{\pi}} \tau^{7/2} f_{7/2} \right) \\ & + \frac{\partial^2 (d_1/\pi)}{\partial \mu^2} \frac{1}{|u|} \left(\frac{1}{\sqrt{\pi}} \tau^{1/2} f_{1/2} - \frac{1}{\sqrt{\pi}} \tau^{3/2} f_{3/2} + \frac{3}{2\sqrt{\pi}} \tau^{5/2} f_{5/2} - \frac{15}{8\sqrt{\pi}} \tau^{7/2} f_{7/2} \right) \\ & = \frac{\partial}{\partial \mu} \left(\tau^{1/2} f_{1/2} \right) \left(-\frac{1}{\sqrt{\pi}} \right) \left(2 - \frac{\partial \eta}{\partial \mu} + \frac{1}{|u|} \frac{\partial (d_1/\pi)}{\partial \mu} \right) \\ & + \frac{\partial}{\partial \mu} \left(\tau^{3/2} f_{3/2} \right) \left(-\frac{1}{2\sqrt{\pi}} \frac{\partial \eta}{\partial \mu} - \frac{1}{2\sqrt{\pi}} \frac{\partial (u^2 \varphi)}{\partial \mu} + \frac{1}{|u|} \frac{1}{\sqrt{\pi}} \frac{\partial (d_1/\pi)}{\partial \mu} + \frac{1}{\sqrt{\pi}} \right) \\ & + \frac{\partial}{\partial \mu} \left(\tau^{5/2} f_{5/2} \right) \left(\frac{3}{4\sqrt{\pi}} \frac{\partial \eta}{\partial \mu} + \frac{3}{4\sqrt{\pi}} \frac{\partial (u^2 \varphi)}{\partial \mu} - \frac{1}{|u|} \frac{3}{2\sqrt{\pi}} \frac{\partial (d_1/\pi)}{\partial \mu} - \frac{3}{2\sqrt{\pi}} \right) \\ & + \frac{\partial}{\partial \mu} \left(\tau^{7/2} f_{7/2} \right) \frac{15}{8\sqrt{\pi}} \left(\frac{1}{|u|} \frac{\partial (d_1/\pi)}{\partial \mu} - \frac{\partial (u^2 \varphi)}{\partial \mu} \right), \end{aligned} \quad (3.2.25)$$

$$\begin{aligned} & \frac{\partial^2 \eta}{\partial \mu^2} \left(1 - \frac{1}{2\sqrt{\pi}} \tau^{1/2} f_{1/2} + \frac{1}{16\sqrt{\pi}} \tau^{3/2} f_{3/2} - \frac{3}{128\sqrt{\pi}} \tau^{5/2} f_{5/2} \right) \\ & + \frac{\partial^2 (u^2 \varphi)}{\partial \mu^2} \left(\frac{1}{4\sqrt{\pi}} \tau^{3/2} f_{3/2} - \frac{3}{32\sqrt{\pi}} \tau^{5/2} f_{5/2} + \frac{15}{256\sqrt{\pi}} \tau^{7/2} f_{7/2} \right) \\ & + \frac{\partial^2 (d_1/\pi)}{\partial \mu^2} \frac{1}{|u|} \left(\frac{1}{2\sqrt{\pi}} \tau^{1/2} f_{1/2} - \frac{5}{16\sqrt{\pi}} \tau^{3/2} f_{3/2} + \frac{5}{128\sqrt{\pi}} \tau^{5/2} f_{5/2} - \frac{15}{256\sqrt{\pi}} \tau^{7/2} f_{7/2} \right) \\ & = \frac{\partial}{\partial \mu} \left(\tau^{1/2} f_{1/2} \right) \left(-\frac{1}{2\sqrt{\pi}} \right) \left(2 - \frac{\partial \eta}{\partial \mu} + \frac{1}{|u|} \frac{\partial (d_1/\pi)}{\partial \mu} \right) \\ & + \frac{\partial}{\partial \mu} \left(\tau^{3/2} f_{3/2} \right) \left(-\frac{1}{16\sqrt{\pi}} \frac{\partial \eta}{\partial \mu} - \frac{1}{4\sqrt{\pi}} \frac{\partial (u^2 \varphi)}{\partial \mu} + \frac{1}{|u|} \frac{5}{16\sqrt{\pi}} \frac{\partial (d_1/\pi)}{\partial \mu} + \frac{1}{8\sqrt{\pi}} \right) \end{aligned}$$

$$\begin{aligned}
& + \frac{\partial}{\partial \mu} \left(\tau^{5/2} f_{5/2} \right) \left(\frac{3}{128\sqrt{\pi}} \frac{\partial \eta}{\partial \mu} + \frac{3}{32\sqrt{\pi}} \frac{\partial(u^2 \varphi)}{\partial \mu} - \frac{1}{|u|} \frac{5}{128\sqrt{\pi}} \frac{\partial(d_1/\pi)}{\partial \mu} - \frac{3}{64\sqrt{\pi}} \right) \\
& + \frac{\partial}{\partial \mu} \left(\tau^{7/2} f_{7/2} \right) \frac{15}{256\sqrt{\pi}} \left(-\frac{\partial(u^2 \varphi)}{\partial \mu} + \frac{1}{|u|} \frac{\partial(d_1/\pi)}{\partial \mu} \right), \tag{3.2.26}
\end{aligned}$$

$$\begin{aligned}
& - \frac{\partial^2 \eta}{\partial \mu^2} \left(\frac{1}{8\sqrt{\pi}} \tau^{1/2} f_{1/2} - \frac{1}{32\sqrt{\pi}} \tau^{3/2} f_{3/2} + \frac{9}{512\sqrt{\pi}} \tau^{5/2} f_{5/2} \right) \\
& + \frac{\partial^2(u^2 \varphi)}{\partial \mu^2} \left(1 + \frac{1}{16\sqrt{\pi}} \tau^{3/2} f_{3/2} - \frac{3}{64\sqrt{\pi}} \tau^{5/2} f_{5/2} + \frac{45}{1024\sqrt{\pi}} \tau^{7/2} f_{7/2} \right) \\
& + \frac{\partial^2(d_1/\pi)}{\partial \mu^2} \frac{1}{|u|} \left(\frac{1}{8\sqrt{\pi}} \tau^{1/2} f_{1/2} - \frac{3}{32\sqrt{\pi}} \tau^{3/2} f_{3/2} + \frac{33}{512\sqrt{\pi}} \tau^{5/2} f_{5/2} - \frac{45}{1024\sqrt{\pi}} \tau^{7/2} f_{7/2} \right) \\
& = \frac{\partial}{\partial \mu} \left(\tau^{1/2} f_{1/2} \right) \left(-\frac{1}{8\sqrt{\pi}} \right) \left(2 - \frac{\partial \eta}{\partial \mu} + \frac{1}{|u|} \frac{\partial(d_1/\pi)}{\partial \mu} \right) \\
& + \frac{\partial}{\partial \mu} \left(\tau^{3/2} f_{3/2} \right) \left(-\frac{1}{32\sqrt{\pi}} \frac{\partial \eta}{\partial \mu} - \frac{1}{16\sqrt{\pi}} \frac{\partial(u^2 \varphi)}{\partial \mu} + \frac{3}{32\sqrt{\pi}} \frac{1}{|u|} \frac{\partial(d_1/\pi)}{\partial \mu} + \frac{3}{16\sqrt{\pi}} \right) \\
& + \frac{\partial}{\partial \mu} \left(\tau^{5/2} f_{5/2} \right) \left(\frac{9}{512\sqrt{\pi}} \frac{\partial \eta}{\partial \mu} + \frac{3}{64\sqrt{\pi}} \frac{\partial(u^2 \varphi)}{\partial \mu} - \frac{1}{|u|} \frac{33}{512\sqrt{\pi}} \frac{\partial(d_1/\pi)}{\partial \mu} - \frac{9}{256\sqrt{\pi}} \right) \\
& + \frac{\partial}{\partial \mu} \left(\tau^{7/2} f_{7/2} \right) \frac{45}{1024\sqrt{\pi}} \left(-\frac{\partial(u^2 \varphi)}{\partial \mu} + \frac{1}{|u|} \frac{\partial(d_1/\pi)}{\partial \mu} \right), \tag{3.2.27}
\end{aligned}$$

$$2\delta \frac{\partial^2 p^b}{\partial \mu^2} + \frac{\partial^2(d_1/\pi)}{\partial \mu^2} = 2 \left(1 - \frac{\partial p^b}{\partial \mu} \right)^2 \delta', \tag{3.2.28}$$

where we have denoted

$$\delta' = \frac{1}{\pi T} \int_{-1}^1 dx \frac{\exp(x/t)/z}{(1 + \exp(x/t)/z)^2} \frac{x}{\sqrt{1-x^2}}. \tag{3.2.29}$$

Solving this set of equations, and omitting all $o(\frac{1}{|u|})$ and $o(\tau^2)$ terms results in that

$$\frac{\partial^2 p^b}{\partial \mu^2} = - \frac{128 f_{-1/2} \left(16\pi - \sqrt{\pi} \tau^{3/2} f_{3/2} \right)}{D_1 \tau^{1/2} \Delta_t^3}. \tag{3.2.30}$$

Differentiating eq. (3.2.21) then gives

$$\frac{\partial^2 p^u}{\partial \mu^2} = \left(1 - \frac{\partial p^b}{\partial \mu} \right)^2 \gamma' - \gamma \frac{\partial^2 p^b}{\partial \mu^2}, \tag{3.2.31}$$

where similarly we have denoted

$$\gamma' = \frac{1}{1 - \frac{\partial p^b}{\partial \mu}} \frac{\partial \gamma}{\partial \mu} = \frac{1}{\pi T} \int_{-1}^1 dx \frac{\exp(x/t)/z}{(1 + \exp(x/t)/z)^2} \frac{1}{\sqrt{1-x^2}}. \quad (3.2.32)$$

3.2.2 Derivatives with respect to B

Having obtained the derivatives of the pressures with respect to chemical potential μ , one can carry out similar calculations regarding magnetic field B . We list the linear equations of first order derivatives as follows,

$$\begin{aligned} & \frac{\partial p^b}{\partial B} - \frac{\partial \eta}{\partial B} \left(\frac{1}{\sqrt{\pi}} \tau^{1/2} f_{1/2} - \frac{1}{2\sqrt{\pi}} \tau^{3/2} f_{3/2} + \frac{3}{4\sqrt{\pi}} \tau^{5/2} f_{5/2} \right) \\ & + \frac{\partial(u^2 \phi)}{\partial B} \left(\frac{1}{2\sqrt{\pi}} \tau^{3/2} f_{3/2} - \frac{3}{4\sqrt{\pi}} \tau^{5/2} f_{5/2} + \frac{15}{8\sqrt{\pi}} \tau^{7/2} f_{7/2} \right) \\ & + \frac{\partial(d_1/\pi)}{\partial B} \frac{1}{|u|} \left(\frac{1}{\sqrt{\pi}} \tau^{1/2} f_{1/2} - \frac{1}{\sqrt{\pi}} \tau^{3/2} f_{3/2} + \frac{3}{2\sqrt{\pi}} \tau^{5/2} f_{5/2} - \frac{15}{8\sqrt{\pi}} \tau^{7/2} f_{7/2} \right) \\ & = 0, \end{aligned} \quad (3.2.33)$$

$$\begin{aligned} & \frac{\partial \eta}{\partial B} \left(1 - \frac{1}{2\sqrt{\pi}} \tau^{1/2} f_{1/2} + \frac{1}{16\sqrt{\pi}} \tau^{3/2} f_{3/2} - \frac{3}{128\sqrt{\pi}} \tau^{5/2} f_{5/2} \right) \\ & + \frac{\partial(u^2 \phi)}{\partial B} \left(\frac{1}{4\sqrt{\pi}} \tau^{3/2} f_{3/2} - \frac{3}{32\sqrt{\pi}} \tau^{5/2} f_{5/2} + \frac{15}{256\sqrt{\pi}} \tau^{7/2} f_{7/2} \right) \\ & + \frac{\partial(d_1/\pi)}{\partial B} \frac{1}{|u|} \left(\frac{1}{2\sqrt{\pi}} \tau^{1/2} f_{1/2} - \frac{5}{16\sqrt{\pi}} \tau^{3/2} f_{3/2} + \frac{5}{128\sqrt{\pi}} \tau^{5/2} f_{5/2} - \frac{15}{256\sqrt{\pi}} \tau^{7/2} f_{7/2} \right) \\ & = 0, \end{aligned} \quad (3.2.34)$$

$$\begin{aligned} & - \frac{\partial \eta}{\partial B} \left(\frac{1}{8\sqrt{\pi}} \tau^{1/2} f_{1/2} - \frac{1}{32\sqrt{\pi}} \tau^{3/2} f_{3/2} + \frac{9}{512\sqrt{\pi}} \tau^{5/2} f_{5/2} \right) \\ & + \frac{\partial(u^2 \phi)}{\partial B} \left(1 + \frac{1}{16\sqrt{\pi}} \tau^{3/2} f_{3/2} - \frac{3}{64\sqrt{\pi}} \tau^{5/2} f_{5/2} + \frac{45}{1024\sqrt{\pi}} \tau^{7/2} f_{7/2} \right) \\ & + \frac{\partial(d_1/\pi)}{\partial B} \frac{1}{|u|} \left(\frac{1}{8\sqrt{\pi}} \tau^{1/2} f_{1/2} - \frac{3}{32\sqrt{\pi}} \tau^{3/2} f_{3/2} + \frac{33}{512\sqrt{\pi}} \tau^{5/2} f_{5/2} - \frac{45}{1024\sqrt{\pi}} \tau^{7/2} f_{7/2} \right) \\ & = 0, \end{aligned} \quad (3.2.35)$$

$$2\delta \frac{\partial p^b}{\partial B} + \frac{\partial(d_1/\pi)}{\partial B} = 2\delta, \quad (3.2.36)$$

where we have used

$$\frac{\partial D_1}{\partial B} = \frac{1}{|u|} \frac{\partial(d_1/\pi)}{\partial B} - \frac{\partial(u^2\varphi)}{\partial B}, \quad (3.2.37)$$

$$\frac{T}{\zeta} \frac{\partial \zeta}{\partial B} = -\frac{\partial \eta}{\partial B} + \frac{1}{|u|} \frac{\partial(d_1/\pi)}{\partial B}. \quad (3.2.38)$$

The solution to this set of equations is expressed as

$$\frac{\partial p^b}{\partial B} = -\frac{16\delta\tau^{1/2}(f_{1/2} - \tau f_{3/2})}{|u|\Delta_t}, \quad (3.2.39)$$

$$\frac{\partial \eta}{\partial B} = -\frac{\delta\tau^{1/2}(8f_{1/2} - 5\tau f_{3/2})}{|u|\Delta_t}, \quad (3.2.40)$$

$$\frac{\partial(u^2\varphi)}{\partial B} = -\frac{2\delta\tau^{1/2}f_{1/2}}{|u|\Delta_t}, \quad (3.2.41)$$

$$\frac{\partial(d_1/\pi)}{\partial B} = 2\delta + \frac{32\delta^2\tau^{1/2}(f_{1/2} - \tau f_{3/2})}{|u|\Delta_t}. \quad (3.2.42)$$

The linear equations for the second order derivatives are given by

$$\begin{aligned} & \frac{\partial^2 p^b}{\partial B^2} - \frac{\partial^2 \eta}{\partial B^2} \left(\frac{1}{\sqrt{\pi}} \tau^{1/2} f_{1/2} - \frac{1}{2\sqrt{\pi}} \tau^{3/2} f_{3/2} + \frac{3}{4\sqrt{\pi}} \tau^{5/2} f_{5/2} \right) \\ & + \frac{\partial^2(u^2\varphi)}{\partial B^2} \left(\frac{1}{2\sqrt{\pi}} \tau^{3/2} f_{3/2} - \frac{3}{4\sqrt{\pi}} \tau^{5/2} f_{5/2} + \frac{15}{8\sqrt{\pi}} \tau^{7/2} f_{7/2} \right) \\ & + \frac{\partial^2(d_1/\pi)}{\partial B^2} \frac{1}{|u|} \left(\frac{1}{\sqrt{\pi}} \tau^{1/2} f_{1/2} - \frac{1}{\sqrt{\pi}} \tau^{3/2} f_{3/2} + \frac{3}{2\sqrt{\pi}} \tau^{5/2} f_{5/2} - \frac{15}{8\sqrt{\pi}} \tau^{7/2} f_{7/2} \right) \\ & = \frac{\partial}{\partial B} \left(\tau^{1/2} f_{1/2} \right) \left(-\frac{1}{\sqrt{\pi}} \right) \left(-\frac{\partial \eta}{\partial B} + \frac{1}{|u|} \frac{\partial(d_1/\pi)}{\partial B} \right) \\ & + \frac{\partial}{\partial B} \left(\tau^{3/2} f_{3/2} \right) \left(-\frac{1}{2\sqrt{\pi}} \frac{\partial \eta}{\partial B} - \frac{1}{2\sqrt{\pi}} \frac{\partial(u^2\varphi)}{\partial B} + \frac{1}{|u|} \frac{1}{\sqrt{\pi}} \frac{\partial(d_1/\pi)}{\partial B} \right) \\ & + \frac{\partial}{\partial B} \left(\tau^{5/2} f_{5/2} \right) \left(\frac{3}{4\sqrt{\pi}} \frac{\partial \eta}{\partial B} + \frac{3}{4\sqrt{\pi}} \frac{\partial(u^2\varphi)}{\partial B} - \frac{1}{|u|} \frac{3}{2\sqrt{\pi}} \frac{\partial(d_1/\pi)}{\partial B} \right) \\ & + \frac{\partial}{\partial B} \left(\tau^{7/2} f_{7/2} \right) \frac{15}{8\sqrt{\pi}} \left(\frac{1}{|u|} \frac{\partial(d_1/\pi)}{\partial B} - \frac{\partial(u^2\varphi)}{\partial B} \right), \end{aligned} \quad (3.2.43)$$

$$\begin{aligned}
& \frac{\partial^2 \eta}{\partial B^2} \left(1 - \frac{1}{2\sqrt{\pi}} \tau^{1/2} f_{1/2} + \frac{1}{16\sqrt{\pi}} \tau^{3/2} f_{3/2} - \frac{3}{128\sqrt{\pi}} \tau^{5/2} f_{5/2} \right) \\
& + \frac{\partial^2(u^2 \varphi)}{\partial B^2} \left(\frac{1}{4\sqrt{\pi}} \tau^{3/2} f_{3/2} - \frac{3}{32\sqrt{\pi}} \tau^{5/2} f_{5/2} + \frac{15}{256\sqrt{\pi}} \tau^{7/2} f_{7/2} \right) \\
& + \frac{\partial^2(d_1/\pi)}{\partial B^2} \frac{1}{|u|} \left(\frac{1}{2\sqrt{\pi}} \tau^{1/2} f_{1/2} - \frac{5}{16\sqrt{\pi}} \tau^{3/2} f_{3/2} + \frac{5}{128\sqrt{\pi}} \tau^{5/2} f_{5/2} - \frac{15}{256\sqrt{\pi}} \tau^{7/2} f_{7/2} \right) \\
& = \frac{\partial}{\partial B} \left(\tau^{1/2} f_{1/2} \right) \left(-\frac{1}{2\sqrt{\pi}} \right) \left(-\frac{\partial \eta}{\partial B} + \frac{1}{|u|} \frac{\partial(d_1/\pi)}{\partial B} \right) \\
& + \frac{\partial}{\partial B} \left(\tau^{3/2} f_{3/2} \right) \left(-\frac{1}{16\sqrt{\pi}} \frac{\partial \eta}{\partial B} - \frac{1}{4\sqrt{\pi}} \frac{\partial(u^2 \varphi)}{\partial B} + \frac{1}{|u|} \frac{5}{16\sqrt{\pi}} \frac{\partial(d_1/\pi)}{\partial B} \right) \\
& + \frac{\partial}{\partial B} \left(\tau^{5/2} f_{5/2} \right) \left(\frac{3}{128\sqrt{\pi}} \frac{\partial \eta}{\partial B} + \frac{3}{32\sqrt{\pi}} \frac{\partial(u^2 \varphi)}{\partial B} - \frac{1}{|u|} \frac{5}{128\sqrt{\pi}} \frac{\partial(d_1/\pi)}{\partial B} \right) \\
& + \frac{\partial}{\partial B} \left(\tau^{7/2} f_{7/2} \right) \frac{15}{256\sqrt{\pi}} \left(-\frac{\partial(u^2 \varphi)}{\partial B} + \frac{1}{|u|} \frac{\partial(d_1/\pi)}{\partial B} \right), \tag{3.2.44}
\end{aligned}$$

$$\begin{aligned}
& - \frac{\partial^2 \eta}{\partial B^2} \left(\frac{1}{8\sqrt{\pi}} \tau^{1/2} f_{1/2} - \frac{1}{32\sqrt{\pi}} \tau^{3/2} f_{3/2} + \frac{9}{512\sqrt{\pi}} \tau^{5/2} f_{5/2} \right) \\
& + \frac{\partial^2(u^2 \varphi)}{\partial B^2} \left(1 + \frac{1}{16\sqrt{\pi}} \tau^{3/2} f_{3/2} - \frac{3}{64\sqrt{\pi}} \tau^{5/2} f_{5/2} + \frac{45}{1024\sqrt{\pi}} \tau^{7/2} f_{7/2} \right) \\
& + \frac{\partial^2(d_1/\pi)}{\partial B^2} \frac{1}{|u|} \left(\frac{1}{8\sqrt{\pi}} \tau^{1/2} f_{1/2} - \frac{3}{32\sqrt{\pi}} \tau^{3/2} f_{3/2} + \frac{33}{512\sqrt{\pi}} \tau^{5/2} f_{5/2} - \frac{45}{1024\sqrt{\pi}} \tau^{7/2} f_{7/2} \right) \\
& = \frac{\partial}{\partial B} \left(\tau^{1/2} f_{1/2} \right) \left(-\frac{1}{8\sqrt{\pi}} \right) \left(-\frac{\partial \eta}{\partial B} + \frac{1}{|u|} \frac{\partial(d_1/\pi)}{\partial B} \right) \\
& + \frac{\partial}{\partial B} \left(\tau^{3/2} f_{3/2} \right) \left(-\frac{1}{32\sqrt{\pi}} \frac{\partial \eta}{\partial B} - \frac{1}{16\sqrt{\pi}} \frac{\partial(u^2 \varphi)}{\partial B} + \frac{3}{32\sqrt{\pi}} \frac{1}{|u|} \frac{\partial(d_1/\pi)}{\partial B} \right) \\
& + \frac{\partial}{\partial B} \left(\tau^{5/2} f_{5/2} \right) \left(\frac{9}{512\sqrt{\pi}} \frac{\partial \eta}{\partial B} + \frac{3}{64\sqrt{\pi}} \frac{\partial(u^2 \varphi)}{\partial B} - \frac{1}{|u|} \frac{33}{512\sqrt{\pi}} \frac{\partial(d_1/\pi)}{\partial B} \right) \\
& + \frac{\partial}{\partial B} \left(\tau^{7/2} f_{7/2} \right) \frac{45}{1024\sqrt{\pi}} \left(-\frac{\partial(u^2 \varphi)}{\partial B} + \frac{1}{|u|} \frac{\partial(d_1/\pi)}{\partial B} \right), \tag{3.2.45}
\end{aligned}$$

$$2\delta \frac{\partial^2 p^b}{\partial B^2} + \frac{\partial^2(d_1/\pi)}{\partial B^2} = 2 \left(1 - \frac{\partial p^b}{\partial B} \right)^2 \delta'. \tag{3.2.46}$$

It is easy to write down the solution of interest

$$\begin{aligned} \frac{\partial^2 p^b}{\partial B^2} = & - \frac{16\delta'\tau^{1/2}(f_{1/2} - \tau f_{3/2})}{|u|\Delta_t} \\ & - \frac{4096\pi\delta^2 f_{-1/2} \left(16\sqrt{\pi}f_{1/2} - 8\tau^{1/2}f_{1/2}^2 - 5\tau^{3/2}f_{1/2}f_{3/2} - 4\sqrt{\pi}\tau f_{3/2} \right)}{u^2 D_1 \tau^{1/2} (4f_{1/2} - \tau f_{3/2}) \Delta_t^4}. \end{aligned} \quad (3.2.47)$$

The derivatives of p^u with respect to B are written as

$$\frac{\partial p^u}{\partial B} = \left(1 - \frac{\partial p^b}{\partial B}\right) \gamma, \quad (3.2.48)$$

$$\frac{\partial^2 p^u}{\partial B^2} = \left(1 - \frac{\partial p^b}{\partial B}\right)^2 \gamma' - \gamma \frac{\partial^2 p^b}{\partial B^2}. \quad (3.2.49)$$

3.2.3 Summary for Thermodynamic Quantities

As a consequence, we summarize the results for the thermodynamic quantities here

$$n = \frac{\partial p^u}{\partial \mu} + \frac{\partial p^b}{\partial \mu} = \gamma + (1 - \gamma) \frac{\partial p^b}{\partial \mu}, \quad (3.2.50)$$

$$m = \frac{1}{2} \left(\frac{\partial p^u}{\partial B} + \frac{\partial p^b}{\partial B} \right) = \frac{1}{2} \left[\gamma + (1 - \gamma) \frac{\partial p^b}{\partial B} \right], \quad (3.2.51)$$

$$\kappa_c = \frac{\partial^2 p^u}{\partial \mu^2} + \frac{\partial^2 p^b}{\partial \mu^2} = \left(1 - \frac{\partial p^b}{\partial \mu}\right)^2 \gamma' + (1 - \gamma) \frac{\partial^2 p^b}{\partial \mu^2}, \quad (3.2.52)$$

$$\chi_s = \frac{1}{2} \left(\frac{\partial^2 p^u}{\partial B^2} + \frac{\partial^2 p^b}{\partial B^2} \right) = \frac{1}{2} \left[\left(1 - \frac{\partial p^b}{\partial B}\right)^2 \gamma' + (1 - \gamma) \frac{\partial^2 p^b}{\partial B^2} \right], \quad (3.2.53)$$

where the first and second order derivatives of p^b with respect to μ or B are listed as

$$\frac{\partial p^b}{\partial \mu} = - \frac{8\tau^{1/2} (2f_{1/2} - \tau f_{3/2})}{\Delta_t}, \quad (3.2.54)$$

$$\frac{\partial p^b}{\partial B} = - \frac{16\delta\tau^{1/2} (f_{1/2} - \tau f_{3/2})}{|u|\Delta_t}, \quad (3.2.55)$$

$$\frac{\partial^2 p^b}{\partial \mu^2} = - \frac{128f_{-1/2} (16\pi - \sqrt{\pi}\tau^{3/2}f_{3/2})}{D_1 \tau^{1/2} \Delta_t^3}, \quad (3.2.56)$$

$$\begin{aligned} \frac{\partial^2 p^b}{\partial B^2} = & - \frac{16\delta'\tau^{1/2}(f_{1/2} - \tau f_{3/2})}{|u|\Delta_t} \\ & - \frac{4096\pi\delta^2 f_{-1/2} \left(16\sqrt{\pi}f_{1/2} - 8\tau^{1/2}f_{1/2}^2 - 5\tau^{3/2}f_{1/2}f_{3/2} - 4\sqrt{\pi}\tau f_{3/2} \right)}{u^2 D_1 \tau^{1/2} (4f_{1/2} - \tau f_{3/2}) \Delta_t^4}, \end{aligned} \quad (3.2.57)$$

with

$$\Delta_t = 8\sqrt{\pi} - 4\tau^{1/2}f_{1/2} + \tau^{3/2}f_{3/2}, \quad (3.2.58)$$

$$\delta = \frac{1}{\pi} \int_{-1}^1 dx \frac{1}{1 + \exp(x/t)/z} \frac{x}{\sqrt{1-x^2}}, \quad (3.2.59)$$

$$\gamma = \frac{1}{\pi} \int_{-1}^1 dx \frac{1}{1 + \exp(x/t)/z} \frac{1}{\sqrt{1-x^2}}, \quad (3.2.60)$$

$$\gamma' = \frac{1}{\pi T} \int_{-1}^1 dx \frac{\exp(x/t)/z}{(1 + \exp(x/t)/z)^2} \frac{1}{\sqrt{1-x^2}}, \quad (3.2.61)$$

$$\delta' = \frac{1}{\pi T} \int_{-1}^1 dx \frac{\exp(x/t)/z}{(1 + \exp(x/t)/z)^2} \frac{x}{\sqrt{1-x^2}}. \quad (3.2.62)$$

3.3 Iterative Solution

Although the expressions for the thermodynamic quantities are obtained, they are implicit and being coupled with other equations. In principle, the explicit results can be accessible through lengthy iteration, which is the focus of this section. We will try our best to keep the mathematical rigour in our calculations, however, occasionally it is not as rigorous as expected.

As a start, we denote $a_{3/2} = -\frac{1}{\sqrt{\pi D_1}}$, and thus approximately the pressure of bound pairs is expressed as

$$p^b = a_{3/2} T^{3/2} f_{3/2}. \quad (3.3.1)$$

Moreover, according to the leading terms in eqs. (3.2.1) to (3.2.3), we know that

$$\eta \approx \frac{1}{2} p^b, \quad u^2 \varphi \approx \frac{1}{8} p^b. \quad (3.3.2)$$

In the low density regime, we have assumed $\frac{\Lambda}{n|u|} \ll 1$, which means that p^b is rather small compared to other quantities. Hence one derives that

$$a_{3/2} = -\frac{1}{\sqrt{\pi}} \left(\frac{d_1}{|u|\pi} - u^2 \varphi \right)^{-1/2} \quad (3.3.3)$$

$$\approx - \left(\frac{d_1}{|u|} - \frac{\pi p^b}{8} \right)^{-1/2} \quad (3.3.4)$$

$$\approx - \sqrt{\frac{|u|}{d_1}} - \frac{p^b}{16\sqrt{\pi}} \left(\frac{|u|\pi}{d_1} \right)^{3/2}, \quad (3.3.5)$$

substituting eq. (3.3.1) into which yields

$$a_{3/2} = - \sqrt{\frac{|u|}{d_1}} + \frac{\pi}{16} \left(\frac{u}{d_1} \right)^2 T^{3/2} f_{3/2}. \quad (3.3.6)$$

Hereafter we denote that

$$d_0 = 2\pi - 4 \int_{-1}^1 dx \frac{\sqrt{1-x^2}}{1 + \exp(x/t)/z_0}, \quad (3.3.7)$$

$$\delta_0 = \frac{1}{\pi} \int_{-1}^1 dx \frac{1}{1 + \exp(x/t)/z_0} \frac{x}{\sqrt{1-x^2}}, \quad (3.3.8)$$

$$\gamma_0 = \frac{1}{\pi} \int_{-1}^1 dx \frac{1}{1 + \exp(x/t)/z_0} \frac{1}{\sqrt{1-x^2}}, \quad (3.3.9)$$

$$\delta'_0 = \frac{1}{\pi T} \int_{-1}^1 dx \frac{\exp(x/t)/z_0}{(1 + \exp(x/t)/z_0)^2} \frac{x}{\sqrt{1-x^2}}, \quad (3.3.10)$$

$$\gamma'_0 = \frac{1}{\pi T} \int_{-1}^1 dx \frac{\exp(x/t)/z_0}{(1 + \exp(x/t)/z_0)^2} \frac{1}{\sqrt{1-x^2}}, \quad (3.3.11)$$

with

$$z_0 = \exp \left(\frac{\mu + 2u + B}{T} \right). \quad (3.3.12)$$

With the above notation, we know that

$$z = z_0 \exp(-a_{3/2} T^{1/2} f_{3/2}), \quad (3.3.13)$$

and then we derive

$$\begin{aligned} d_1 &= 2\pi - 4 \int_{-1}^1 dx \frac{\sqrt{1-x^2}}{1 + \exp(x/t)/(z_0 \exp(-a_{3/2} T^{1/2} f_{3/2}))} \\ &\approx 2\pi - 4 \left\{ \int_{-1}^1 dx \frac{\sqrt{1-x^2}}{1 + \exp(x/t)/z_0} + \int_{-1}^1 dx \frac{\exp(x/t)/z_0}{(1 + \exp(x/t)/z_0)^2} \sqrt{1-x^2} \left[\exp(-a_{3/2} T^{1/2} f_{3/2}) - 1 \right] \right\} \\ &\approx d_0 - 4 \int_{-1}^1 dx \frac{\exp(x/t)/z_0}{(1 + \exp(x/t)/z_0)^2} \sqrt{1-x^2} \left(-a_{3/2} T^{1/2} f_{3/2} \right) \end{aligned}$$

$$\begin{aligned}
&= d_0 - 4a_{3/2}T^{1/2}f_{3/2}t \int_{-1}^1 d\frac{x}{t} \frac{\exp(x/t)}{z_0} \frac{-1}{(1 + \exp(x/t)/z_0)^2} \sqrt{1-x^2} \\
&= d_0 + 2p^b \int_{-1}^1 dx \frac{1}{1 + \exp(x/t)/z_0} \frac{x}{\sqrt{1-x^2}} \\
&= d_0 + 2\pi p^b \delta_0,
\end{aligned} \tag{3.3.14}$$

where we take into consideration that d_1 shows itself in the form of $\frac{d_1}{|u|}$ and $\frac{2\pi p^b}{|u|}$ being tiny, we thus approximately have

$$d_1 \approx d_0. \tag{3.3.15}$$

By a similar procedure, we obtain

$$\delta \approx \delta_0, \quad \delta' \approx \delta'_0, \quad \gamma \approx \gamma_0, \quad \gamma' \approx \gamma'_0. \tag{3.3.16}$$

We turn to $f_{3/2}$, with

$$\begin{aligned}
f_{3/2} &= \text{Li}_{3/2}(-\zeta) \\
&\approx \text{Li}_{3/2} \left(-\exp \left(\frac{2\mu - \frac{p^b}{2} + \frac{d_0}{|u|\pi}}{T} \right) \right) \\
&= \text{Li}_{3/2} \left[-\exp \left(\frac{2\mu + \frac{d_0}{|u|\pi}}{T} \right) \exp \left(-\frac{1}{2} T^{1/2} a_{3/2} f_{3/2} \right) \right] \\
&= \text{Li}_{3/2} \left(-\zeta_0 \exp \left(-\frac{1}{2} T^{1/2} a_{3/2} f_{3/2} \right) \right) \\
&= \sum_{k=0}^{\infty} \frac{(-\zeta_0)^k}{k^{3/2}} \left[\exp \left(-\frac{1}{2} T^{1/2} a_{3/2} f_{3/2} \right) \right]^k \\
&\approx \sum_{k=0}^{\infty} \frac{(-\zeta_0)^k}{k^{3/2}} \left[1 - \frac{1}{2} T^{1/2} a_{3/2} f_{3/2} \right]^k \\
&\approx \sum_{k=0}^{\infty} \frac{(-\zeta_0)^k}{k^{3/2}} \left[1 + k \left(-\frac{1}{2} T^{1/2} a_{3/2} f_{3/2} \right) \right] \\
&\approx \sum_{k=0}^{\infty} \frac{(-\zeta_0)^k}{k^{3/2}} - \frac{1}{2} a_{3/2} T^{1/2} f_{3/2} \sum_{k=0}^{\infty} \frac{(-\zeta_0)^k}{k^{1/2}} \\
&= g_{3/2} - \frac{1}{2} T^{1/2} a_{3/2} f_{3/2} g_{1/2},
\end{aligned} \tag{3.3.17}$$

where we have used the definition of the polylogarithm $\text{Li}_s(x) = \sum_{k=0}^{\infty} \frac{x^k}{k^s}$, and

denoted

$$\zeta_0 = \exp\left(\frac{2\mu + \frac{d_0}{|u|\pi}}{T}\right), \quad g_s = \text{Li}_s(-\zeta_0). \quad (3.3.18)$$

Substituting eq. (3.3.17) into eq. (3.3.6) leads to

$$a_{3/2} \approx -\sqrt{\frac{|u|}{d_0}} + \frac{\pi}{16} \left(\frac{|u|}{d_0}\right)^2 T^{3/2} g_{3/2}. \quad (3.3.19)$$

Inserting this result into eq. (3.3.17) gives

$$f_{3/2} \approx g_{3/2} - \frac{1}{2} \sqrt{\frac{|u|}{d_0}} T^{1/2} f_{3/2} g_{1/2}, \quad (3.3.20)$$

which after one iteration can be approximated as

$$f_{3/2} \approx g_{3/2} - \frac{1}{2} \sqrt{\frac{|u|}{d_0}} T^{1/2} g_{3/2} g_{1/2}. \quad (3.3.21)$$

Similarly, we derive that

$$f_{1/2} \approx g_{1/2} - \frac{1}{2} \sqrt{\frac{|u|}{d_0}} T^{1/2} g_{3/2} g_{-1/2}, \quad (3.3.22)$$

$$f_{-1/2} \approx g_{-1/2} - \frac{1}{2} \sqrt{\frac{|u|}{d_0}} T^{1/2} g_{3/2} g_{-3/2}. \quad (3.3.23)$$

Furthermore, we express D_1 as

$$\begin{aligned} D_1 &= \frac{d_0}{|u|\pi} - \frac{p^b}{8} \\ &\approx \frac{d_0}{|u|\pi} - \frac{1}{8} a_{3/2} T^{3/2} f_{3/2} \\ &\approx \frac{d_0}{|u|\pi} + \frac{1}{8} \sqrt{\frac{|u|}{d_0}} T^{3/2} g_{3/2}. \end{aligned} \quad (3.3.24)$$

Hence, the pressures of bound pairs and unpaired fermions are expressed as

$$p^b = -\frac{T^{3/2}}{\sqrt{\pi D_1}} f_{3/2}, \quad (3.3.25)$$

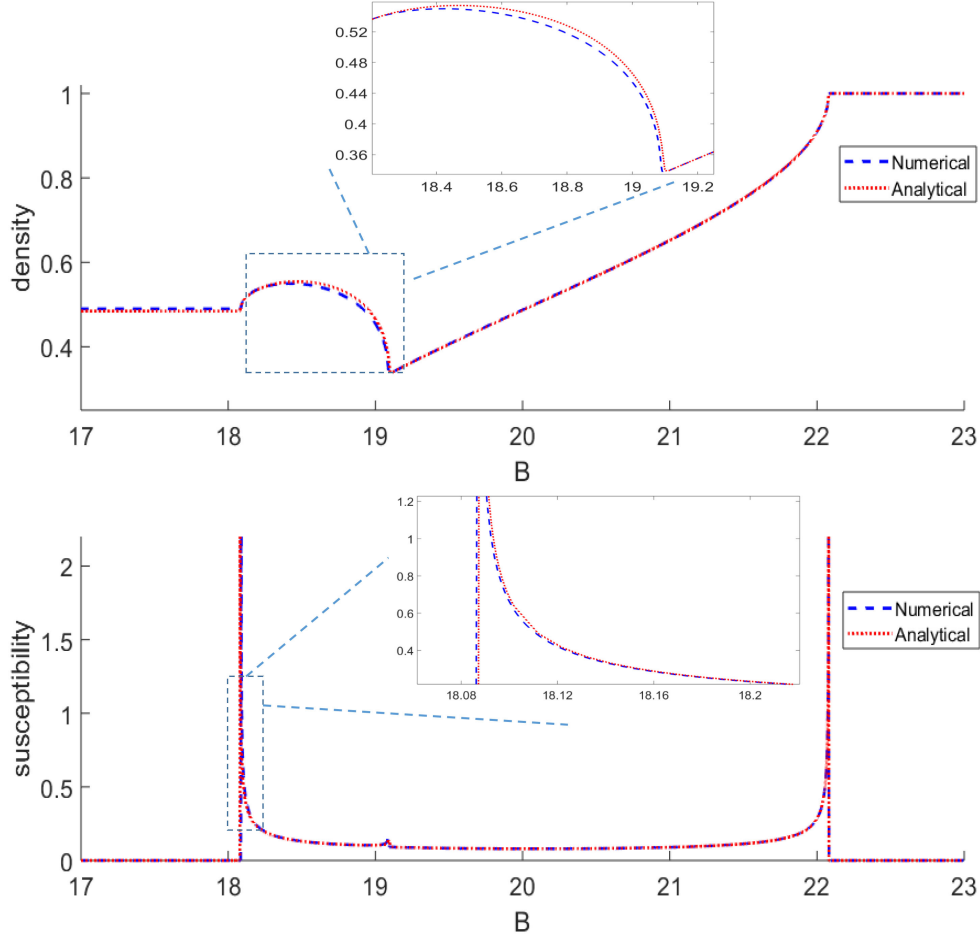


Figure 3.1: A comparison between the analytic results derived in sections 3.2.3 & 3.3 and the numerical results obtained from the TBA equations (2.3.41) to (2.3.43). Here we adopt natural units in the plots. The upper and the lower panels respectively show the density and susceptibility vs magnetic field across phases V, IV, II, III at a fixed chemical potential $\mu = -0.08$, temperature $T = 10^{-4}$ and interaction strength $u = -10$. The sudden changes in the density and susceptibility show subtle scaling behavior near phase transitions.

$$p'' = T \ln \left[1 + \exp \left(\frac{2 - (\mu + 2u + B - p^b)}{T} \right) \right] + \frac{2}{\pi} \int_{-1}^1 dx \frac{\arccos(-x)}{1 + \exp(x/t)/(z_0 \cdot e^{-p^b/T})}. \quad (3.3.26)$$

Now we have arrived at a set of explicit expressions for the thermodynamic quantities, which states that the variables present in section 3.2.3 read

$$D_1 \approx \frac{d_0}{|u|\pi} + \frac{1}{8} \sqrt{\frac{|u|}{d_0}} T^{3/2} g_{3/2}, \quad (3.3.27)$$

$$f_s \approx g_s - \frac{1}{2} \sqrt{\frac{|u|}{d_0}} T^{1/2} g_s g_{s-1}, \quad (3.3.28)$$

and the other variables $d_1, \delta, \gamma, \delta', \gamma'$ have been replaced by $d_0, \delta_0, \gamma_0, \delta'_0, \gamma'_0$. They can be used in fitting experimental data, while their accuracy is demonstrated in fig. 3.1.

3.4 Quantum Criticality

A quantum phase transition in 1D occurs at zero temperature by varying the physical parameters of the system. In general, near a quantum critical point, the model is expected to show universal scaling behaviour in the thermodynamic quantities due to the collective nature of many-body effects [92]. There are five different phases in the ground state of the 1D attractive Hubbard model, which makes itself an ideal model to explore such a universal scale-invariant description on a 1D lattice. The behavior of the thermodynamic quantities is governed by scaling functions with critical exponents in a V-shaped region fanning out to finite temperatures from the quantum critical point.

At very low temperatures, spin fluctuation in the FFLO-like phase is suppressed, as are the bound states of higher k - Λ strings for $|\mu/T| \gg 1$. In this regime, the thermodynamics of the model is governed by a two-component TLL or say two-component Fermi liquid comprising of excess fermions and of hard-core bosonic charge bound states. The leading low-temperature correction to the free energy is given by

$$f \approx f_0 - \frac{\pi T^2}{6} \left(\frac{1}{v^u} + \frac{1}{v^b} \right), \quad (3.4.1)$$

where f_0 is the ground state free energy and v^u (v^b) is the sound velocity of excess unpaired fermions (bound pairs). This result is valid for arbitrary interaction strength.

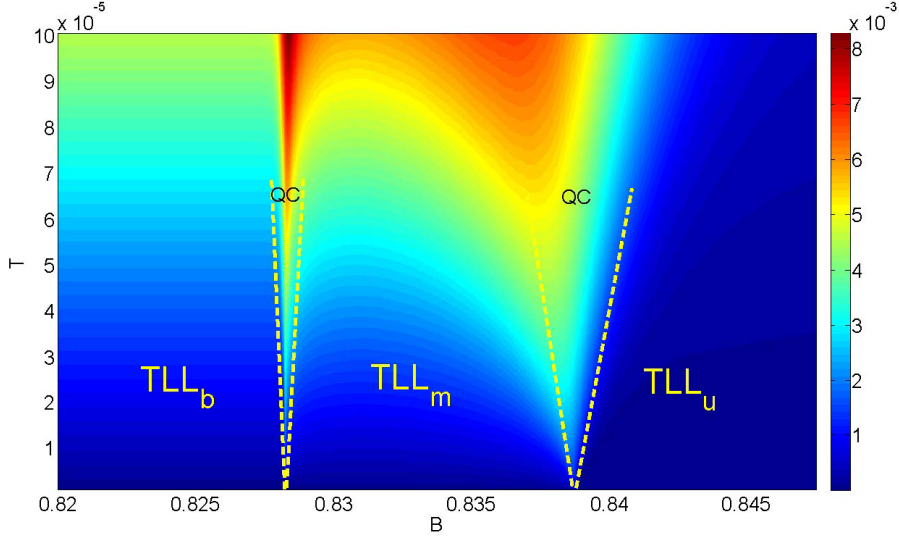


Figure 3.2: Contour plot of entropy vs magnetic field B for the 1D attractive Hubbard model. The numerical calculation is performed by solving the TBA equations (2.3.41) to (2.3.43) with a fixed chemical potential $\mu = -0.828$ and interaction $u = -1$. The crossover temperatures (white dashed lines) fanning out from the critical points separate different TLL phases from the quantum critical regimes. The linear temperature-dependent entropy breaks down when the temperature is greater than these crossover temperatures. Here TLL_u and TLL_b respectively stand for the TLLs of unpaired fermions and bound pairs. TLL_m stands for the two-component TLL of the FFLO-like state.

The TLL validates only in the region below the crossover temperatures, where the entropy or specific heat retains a linear temperature-dependence, see the dashed lines in fig. 3.2. The entropy in the temperature-magnetic field plane displays the visible areas of the critical regions near different critical points. We will derive the scaling functions for the critical regions in this section through expanding the thermodynamic quantities obtained in section 3.2.3 in the regime $|\mu - \mu_c| \ll 1$.

We observe that some integrands in the previous sections contain functions of this kind

$$\frac{1}{1 + \exp(x/a)/b},$$

which originate from part integration and correspond to

$$\frac{1}{1 + \exp(\epsilon(y)/T)},$$

where $\epsilon(y)$ is the dressed energy. This is helpful in later analysis of criticality.

3.4.1 μ_{c1} : Phase I-V

In this quantum critical region, the dressed energy is positive $\kappa > 0$, and thus p^μ , δ , γ , δ' , $\gamma' \approx 0$, $d_1 \approx 2\pi$; $p^b \ll 1$; hence $D_1 \approx \frac{2}{|u|}$. Both the magnetization $\frac{\partial p^b}{\partial B}$ and the susceptibility $\frac{\partial^2 p^b}{\partial B^2}$ vanish due to $\delta', \delta \approx 0$.

Around the critical point, at very low temperature, approximately have

$$n = \frac{\partial p^b}{\partial \mu} = -\frac{8\tau^{1/2}(2f_{1/2} - \tau f_{3/2})}{\Delta_t} \approx -\sqrt{\frac{2|u|}{\pi}} T^{1/2} f_{1/2}, \quad (3.4.2)$$

$$m = \frac{\partial p^b}{\partial B} = -\frac{16\delta\tau^{1/2}(f_{1/2} - \tau f_{3/2})}{|u|\Delta_t} \approx 0, \quad (3.4.3)$$

$$\kappa_c = \frac{\partial^2 p^b}{\partial \mu^2} = -\frac{128f_{-1/2}(16\pi - \sqrt{\pi}\tau^{3/2}f_{3/2})}{D_1\tau^{1/2}\Delta_t^3} \approx -2\sqrt{\frac{2|u|}{\pi}} T^{-1/2} f_{-1/2} \quad (3.4.4)$$

$$\chi_s = \frac{\partial^2 p^b}{\partial B^2} \approx 0. \quad (3.4.5)$$

Furthermore, we have

$$\begin{aligned} f_{1/2} &= \text{Li}_{1/2}(-\zeta) \approx \text{Li}_{1/2}\left(-\exp\left(\frac{2\mu - p^b/2 + 2/c}{T}\right)\right) \\ &= \frac{-1}{\Gamma(\frac{1}{2})} \int_0^\infty dx \frac{x^{-1/2}}{1 + e^x / \exp\left(\frac{2\mu - p^b/2 + 2/c}{T}\right)} \\ &= \frac{-1}{\Gamma(\frac{1}{2})} \int_0^\infty dx \frac{x^{-1/2}}{1 + \exp(x - \frac{2\mu_{c1} - p^b/2 + 2/c}{T}) / \exp\left(\frac{2(\mu - \mu_{c1})}{T}\right)} \\ &\approx \frac{-1}{\Gamma(\frac{1}{2})} \int_0^\infty dx \frac{x^{-1/2}}{1 + e^x / \exp\left(\frac{2(\mu - \mu_{c1})}{T}\right)} \\ &= \text{Li}_{1/2}\left(-\exp\left(\frac{2\mu - 2\mu_{c1}}{T}\right)\right), \end{aligned} \quad (3.4.6)$$

where we have let

$$\exp\left(-\frac{2\mu_{c1} - \frac{p^b}{2} + \frac{2}{|u|}}{T}\right) \approx 1.$$

We here make an explanation of the above approximation, as we have stated in the beginning of this section $e^x / \exp\left(\frac{2\mu - p^b/2 + 2/c}{T}\right)$ corresponds to $\exp(\epsilon'_1/T)$, and the phase transition takes place if μ or B drives this dressed energy function to the situation where $\epsilon'_1(\Lambda = 0, \mu = \mu_c, B, T = 0) = 0$ or $\epsilon'_1(\Lambda = 0, \mu, B = B_c, T = 0) = 0$; obviously $x = 0$ corresponds to $\Lambda = 0$, and then we expand $\epsilon'_1(\Lambda = 0, \mu, B, 0 < T \ll$

$1)/T \approx 0$ around $|\mu - \mu_c| \ll T$, i.e., $\epsilon'_1(\Lambda = 0, \mu, B)/T \approx -(2\mu - \frac{p^b}{2} + \frac{2}{|u|})/T \approx -2(\mu - \mu_c)/T - (2\mu_c - \frac{p^b}{2} + \frac{2}{|u|})/T \approx 0$, which implies that $-(2\mu_c - \frac{p^b}{2} + \frac{2}{|u|})/T$ is higher order term; therefore we keep $-2(\mu - \mu_c)/T$ and thus $\exp(-\frac{2\mu_c - p^b/2 + 2/c}{T}) \approx 1$.

Reasonably, one may conjecture that

$$f_{-1/2} = \text{Li}_{-1/2} \left(-\exp \left(\frac{2\mu - 2\mu_{c1}}{T} \right) \right). \quad (3.4.7)$$

We give a sketch of the proof. We have noted that

$$\frac{\partial}{\partial z} \text{Li}_s(z) = \frac{1}{z} \text{Li}_{s-1}(z), \quad (3.4.8)$$

$$\text{Li}_{1/2}(-z) = \frac{-1}{\Gamma(\frac{1}{2})} \int_0^\infty dx \frac{x^{1/2}}{1 + e^{x/z}}. \quad (3.4.9)$$

It is straightforward to derive that

$$\text{Li}_{-1/2}(-z) = z \frac{\partial}{\partial z} \text{Li}_{1/2} = \frac{-1}{\Gamma(\frac{1}{2})} \int_0^\infty dx \frac{x^{-1/2} e^{x/z}}{(1 + e^{x/z})^2}. \quad (3.4.10)$$

By the help of this alternative definition for $\text{Li}_{-1/2}(x)$, and through a similar procedure for dealing with $f_{1/2}$ beforehand, we obtain

$$\begin{aligned} f_{-1/2} &= \text{Li}_{-1/2}(-\zeta) \approx \text{Li}_{-1/2} \left(-\exp \left(\frac{2\mu - p^b/2 + 2/c}{T} \right) \right) \\ &= \frac{-1}{\Gamma(\frac{1}{2})} \int_0^\infty dx \frac{x^{-1/2} e^{x/\exp \left(\frac{2\mu - p^b/2 + 2/c}{T} \right)}}{\left(1 + e^{x/\exp \left(\frac{2\mu - p^b/2 + 2/c}{T} \right)} \right)^2} \\ &= \frac{-1}{\Gamma(\frac{1}{2})} \int_0^\infty dx \frac{x^{-1/2} e^{x/\exp \left(\frac{2\mu - 2\mu_{c1}}{T} \right)}}{\left(1 + e^{x/\exp \left(\frac{2\mu - 2\mu_{c1}}{T} \right)} \right)^2} \\ &= \text{Li}_{-1/2} \left(-\exp \left(\frac{2\mu - 2\mu_{c1}}{T} \right) \right). \end{aligned} \quad (3.4.11)$$

Consequently, the critical behaviour of thermodynamic quantities between phases I and V is listed as follows,

$$n = -\sqrt{\frac{2|u|}{\pi}} T^{1/2} \text{Li}_{1/2} \left(-\exp \left(\frac{2\mu - 2\mu_{c1}}{T} \right) \right), \quad (3.4.12)$$

$$m \approx 0, \quad (3.4.13)$$

$$\kappa_c = -2\sqrt{\frac{2|u|}{\pi}} T^{-1/2} \text{Li}_{-1/2} \left(-\exp \left(\frac{2\mu - 2\mu_{c1}}{T} \right) \right) \quad (3.4.14)$$

$$\chi_s \approx 0. \quad (3.4.15)$$

3.4.2 μ_{c2} : Phase I-II

In this critical region, $\epsilon'_1 > 0$, and thus p^b , $\frac{\partial p^b}{\partial \mu}$, $\frac{\partial p^b}{\partial B} \approx 0$.

We start the derivation from n , with

$$n = \frac{\partial p^u}{\partial \mu} = \gamma = \frac{1}{\pi} \int_{-1}^1 dx \frac{1/\sqrt{1-x^2}}{1 + \exp(x/t)/z}. \quad (3.4.16)$$

Similarly, here $\exp(x/t)/z$ originates from $\exp(\kappa(k)/T)$; $x = -1$ corresponds to $\kappa(k=0) = -2 - \mu_{c2} - 2u - B$; once $\kappa(k) > 0$, the integrand decays exponentially, and a phase transition occurs if $\kappa(k=0, \mu = \mu_c, B, T = 0) = 0$ or $\kappa(k=0, \mu = \mu_c, B = B_c, T = 0) = 0$, hence we approximately obtain

$$\begin{aligned} \exp(x/t)/z &= \exp \left(\frac{2x - \mu - 2u - B}{T} \right) \\ &= \exp \left(\frac{2x - (\mu - \mu_{c2}) - \mu_{c2} - 2u - B}{T} \right) \\ &\approx \exp \left(\frac{2x + 2}{T} \right) / \exp \left(\frac{\mu - \mu_{c2}}{T} \right), \end{aligned} \quad (3.4.17)$$

where, due to similar analysis to that around μ_{c1} , we have let

$$\exp \left(\frac{-2 - \mu_{c2} - 2u - B}{T} \right) \approx 1.$$

Going on with the calculation results in

$$\begin{aligned} n &\approx \int_{-1}^1 dx \frac{1/\sqrt{1-x^2}}{1 + \exp \left(\frac{2x+2}{T} \right) / \exp \left(\frac{\mu - \mu_{c2}}{T} \right)} \\ &= \frac{1}{\pi} \int_0^2 dy \frac{1}{1 + \exp \left(\frac{2y}{T} \right) / \exp \left(\frac{\mu - \mu_{c2}}{T} \right)} \frac{1}{\sqrt{1-(y-1)^2}} \\ &\approx \int_0^\infty dy \frac{1}{1 + \exp \left(\frac{2y}{T} \right) / \exp \left(\frac{\mu - \mu_{c2}}{T} \right)} \frac{1}{\sqrt{2y}} \\ &= \frac{1}{2\pi} T^{1/2} \int_0^\infty dq \frac{q^{-1/2}}{1 + e^q / \exp \left(\frac{\mu - \mu_{c2}}{T} \right)} \end{aligned}$$

$$= -\frac{1}{2\sqrt{\pi}} T^{1/2} \text{Li}_{1/2} \left(-\exp \left(\frac{\mu - \mu_{c2}}{T} \right) \right), \quad (3.4.18)$$

where we enlarge the upper limit of integration from 2 to ∞ and only retain the leading term in the expansion of $\frac{1}{\sqrt{y(2-y)}}$ due to this integral mainly relying on a narrow interval near $y = 0$.

Considering $m = \frac{\gamma}{2}$ in phase II, it is easy to see that

$$m = -\frac{1}{4\sqrt{\pi}} T^{1/2} \text{Li}_{1/2} \left(-\exp \left(\frac{\mu - \mu_{c2}}{T} \right) \right). \quad (3.4.19)$$

We turn to the compressibility, for which one should keep in mind the equality for $\text{Li}_{-1/2}(z)$ shown in eq. (3.4.10). We then carry out the derivation for compressibility,

$$\begin{aligned} \kappa_c = \gamma' &= \frac{1}{\pi T} \int_{-1}^1 dx \frac{\exp(x/t)/z}{(1 + \exp(x/t)/z)^2} \frac{1}{\sqrt{1-x^2}} \\ &\approx \frac{1}{\pi T} \int_{-1}^1 dx \frac{\exp\left(\frac{2x+2}{T}\right) / \exp\left(\frac{\mu-\mu_{c2}}{T}\right)}{(1 + \exp\left(\frac{2x+2}{T}\right) / \exp\left(\frac{\mu-\mu_{c2}}{T}\right))^2} \frac{1}{\sqrt{1-x^2}} \\ &\approx \frac{1}{\pi T} \int_0^2 dy \frac{\exp(y/t) / \exp\left(\frac{\mu-\mu_{c2}}{T}\right)}{(1 + \exp(y/t) / \exp\left(\frac{\mu-\mu_{c2}}{T}\right))^2} \frac{1}{\sqrt{y(2-y)}} \\ &\approx \frac{1}{\pi T} \int_0^\infty dy \frac{\exp(y/t) / \exp\left(\frac{\mu-\mu_{c2}}{T}\right)}{(1 + \exp(y/t) / \exp\left(\frac{\mu-\mu_{c2}}{T}\right))^2} \frac{1}{\sqrt{2y}} \\ &= \frac{T^{-1/2}}{2\pi} \int_0^\infty dq q^{-1/2} \frac{e^q / \exp\left(\frac{\mu-\mu_{c2}}{T}\right)}{(1 + e^q / \exp\left(\frac{\mu-\mu_{c2}}{T}\right))^2} \\ &= -\frac{1}{2\sqrt{\pi}} T^{-1/2} \text{Li}_{-1/2} \left(-\exp \left(\frac{\mu - \mu_{c2}}{T} \right) \right). \end{aligned} \quad (3.4.20)$$

Similar to the treatment of m , one can see that

$$\chi_c = \frac{1}{2} \kappa_c = -\frac{1}{4\sqrt{\pi}} T^{-1/2} \text{Li}_{-1/2} \left(-\exp \left(\frac{\mu - \mu_{c2}}{T} \right) \right). \quad (3.4.21)$$

The critical behaviour of thermodynamic quantities between phase I and II is summarised as follows

$$n = -\frac{1}{2\sqrt{\pi}} T^{1/2} \text{Li}_{1/2} \left(-\exp \left(\frac{\mu - \mu_{c2}}{T} \right) \right), \quad (3.4.22)$$

$$m = -\frac{1}{4\sqrt{\pi}} T^{1/2} \text{Li}_{1/2} \left(-\exp \left(\frac{\mu - \mu_{c2}}{T} \right) \right), \quad (3.4.23)$$

$$\kappa_c = -\frac{1}{2\sqrt{\pi}} T^{-1/2} \text{Li}_{-1/2} \left(-\exp \left(\frac{\mu - \mu_{c2}}{T} \right) \right), \quad (3.4.24)$$

$$\chi_c = -\frac{1}{4\sqrt{\pi}} T^{-1/2} \text{Li}_{-1/2} \left(-\exp \left(\frac{\mu - \mu_{c2}}{T} \right) \right). \quad (3.4.25)$$

3.4.3 μ_{c3} : Phase II-III

Similar to the critical region between phase I and II, here we have $\varepsilon'_1(\Lambda) > 0$, and thus p^b , $\frac{\partial p^b}{\partial \mu}$, $\frac{\partial p^b}{\partial B} \approx 0$; while one should notice that the phase transition between phase II and III occurs if μ and B drives the dressed energy to the situation where $\kappa(k = \pi, \mu = \mu_c, B, T = 0) = 0$ or $\kappa(k = \pi, \mu = \mu, B = B_c, T = 0) = 0$.

As a start, the density is expressed as

$$\begin{aligned} n = \frac{\partial p^\mu}{\partial \mu} &= \gamma = \frac{1}{\pi} \int_{-1}^1 dx \frac{1/\sqrt{1-x^2}}{1 + \exp(x/t)/z} \\ &= 1 - \frac{1}{\pi} \int_{-1}^1 dx \frac{\exp(x/t)/z}{1 + \exp(x/t)/z} \frac{1}{\sqrt{1-x^2}} \\ &\approx 1 - \frac{1}{\pi} \int_{-1}^1 dx \frac{\exp\left(\frac{2x-2}{T}\right) / \exp\left(\frac{\mu-\mu_{c3}}{T}\right)}{1 + \exp\left(\frac{2x-2}{T}\right) / \exp\left(\frac{\mu-\mu_{c3}}{T}\right)} \frac{1}{\sqrt{1-x^2}} \\ &\approx 1 - \frac{1}{\pi} \int_{-2}^0 dq \frac{\exp(q/t) / \exp\left(\frac{\mu-\mu_{c3}}{T}\right)}{1 + \exp(q/t) / \exp\left(\frac{\mu-\mu_{c3}}{T}\right)} \frac{1}{\sqrt{1-(q+1)^2}} \\ &= 1 - \frac{1}{\pi} \int_0^2 dy \frac{\exp(-y/t) / \exp\left(\frac{\mu-\mu_{c3}}{T}\right)}{1 + \exp(-y/t) / \exp\left(\frac{\mu-\mu_{c3}}{T}\right)} \frac{1}{\sqrt{2y-y^2}} \\ &= 1 - \frac{1}{\pi} \int_0^2 dy \frac{1}{1 + \exp(y/t) / \exp\left(-\frac{\mu-\mu_{c3}}{T}\right)} \frac{1}{\sqrt{2y-y^2}}, \end{aligned} \quad (3.4.26)$$

where we have let

$$\exp(x/t) \approx \exp\left(\frac{2x-2}{T}\right) / \exp\left(\frac{\mu-\mu_{c3}}{T}\right)$$

due to the similar reason for the approximation in eq. (3.4.6).

Note that $\exp(y/t) / \exp\left(-\frac{\mu-\mu_{c3}}{T}\right)$ corresponds to $\exp(-\kappa/T)$, $y = 0$ corresponds to $x = 1$, and the integral mainly relies on a narrow interval near $y = 0$, we approximately derive that

$$\begin{aligned} n &\approx 1 - \frac{1}{\pi} \int_0^2 dy \frac{1}{1 + \exp(y/t) / \exp\left(-\frac{\mu-\mu_{c3}}{T}\right)} \frac{1}{\sqrt{2y}}, \\ &= 1 - \frac{T^{1/2}}{2\pi} \int_0^{4/T} dx \frac{x^{-1/2}}{1 + e^x / \exp\left(-\frac{\mu-\mu_{c3}}{T}\right)} \end{aligned}$$

$$\begin{aligned}
&\approx 1 - \frac{T^{1/2}}{2\pi} \int_0^\infty dx \frac{x^{-1/2}}{1 + e^x / \exp\left(-\frac{\mu - \mu_{c3}}{T}\right)} \\
&= 1 + \frac{1}{2\sqrt{\pi}} T^{1/2} \text{Li}_{1/2} \left(-\exp\left(-\frac{\mu - \mu_{c3}}{T}\right) \right). \tag{3.4.27}
\end{aligned}$$

In phase II and III, we know that $m = \frac{1}{2}n$, and thus in the quantum critical region between these phases, the magnetization follows as

$$m = \frac{1}{2} + \frac{1}{4\sqrt{\pi}} T^{1/2} \text{Li}_{1/2} \left(-\exp\left(-\frac{\mu - \mu_{c3}}{T}\right) \right). \tag{3.4.28}$$

The derivation for κ_c is loosely analogous to that for n ,

$$\begin{aligned}
\kappa_c = \gamma' &= \frac{1}{\pi T} \int_{-1}^1 dx \frac{\exp(x/t)/z}{(1 + \exp(x/t)/z)^2} \frac{1}{\sqrt{1-x^2}} \\
&\approx \frac{1}{\pi T} \int_{-1}^1 dx \frac{\exp\left(\frac{x-1}{t}\right) / \exp\left(\frac{\mu - \mu_{c3}}{T}\right)}{(1 + \exp\left(\frac{x-1}{t}\right) / \exp\left(\frac{\mu - \mu_{c3}}{T}\right))^2} \frac{1}{\sqrt{1-x^2}} \\
&= \frac{1}{\pi T} \int_{-2}^0 dy \frac{\exp\left(\frac{y}{t}\right) / \exp\left(\frac{\mu - \mu_{c3}}{T}\right)}{(1 + \exp\left(\frac{y}{t}\right) / \exp\left(\frac{\mu - \mu_{c3}}{T}\right))^2} \frac{1}{\sqrt{1-(1+y)^2}} \\
&= \frac{1}{\pi T} \int_0^2 dq \frac{\exp\left(-\frac{q}{t}\right) / \exp\left(\frac{\mu - \mu_{c3}}{T}\right)}{(1 + \exp\left(-\frac{q}{t}\right) / \exp\left(\frac{\mu - \mu_{c3}}{T}\right))^2} \frac{1}{\sqrt{1-(1-q)^2}} \\
&= \frac{1}{\pi T} \int_0^2 dq \frac{\exp\left(\frac{q}{t}\right) / \exp\left(-\frac{\mu - \mu_{c3}}{T}\right)}{(1 + \exp\left(\frac{q}{t}\right) / \exp\left(-\frac{\mu - \mu_{c3}}{T}\right))^2} \frac{1}{\sqrt{1-(1-q)^2}} \\
&\approx \frac{1}{\pi T} \int_0^2 dq \frac{\exp\left(\frac{q}{t}\right) / \exp\left(-\frac{\mu - \mu_{c3}}{T}\right)}{(1 + \exp\left(\frac{q}{t}\right) / \exp\left(-\frac{\mu - \mu_{c3}}{T}\right))^2} \frac{1}{\sqrt{2q}} \\
&= \frac{T^{-1/2}}{2\pi} \int_0^{2/t} dx \frac{e^x / \exp\left(-\frac{\mu - \mu_{c3}}{T}\right)}{(1 + e^x / \exp\left(-\frac{\mu - \mu_{c3}}{T}\right))^2} \frac{1}{\sqrt{x}} \\
&\approx \frac{T^{-1/2}}{2\pi} \int_0^\infty dx \frac{e^x / \exp\left(-\frac{\mu - \mu_{c3}}{T}\right)}{(1 + e^x / \exp\left(-\frac{\mu - \mu_{c3}}{T}\right))^2} \frac{1}{\sqrt{x}} \\
&= -\frac{1}{2\sqrt{\pi}} T^{-1/2} \text{Li}_{-1/2} \left(-\exp\left(-\frac{\mu - \mu_{c3}}{T}\right) \right). \tag{3.4.29}
\end{aligned}$$

Similar to the treatment of the magnetization, the susceptibility is expressed as

$$\chi_s = \frac{1}{2} \gamma' = -\frac{1}{4\sqrt{\pi}} T^{-1/2} \text{Li}_{-1/2} \left(-\exp\left(-\frac{\mu - \mu_{c3}}{T}\right) \right). \tag{3.4.30}$$

The critical behaviour of thermodynamic quantities between phase II and III is as follows

$$n = 1 + \frac{1}{2\sqrt{\pi}} T^{1/2} \text{Li}_{1/2} \left(-\exp \left(-\frac{\mu - \mu_{c3}}{T} \right) \right), \quad (3.4.31)$$

$$m = \frac{1}{2} + \frac{1}{4\sqrt{\pi}} T^{1/2} \text{Li}_{1/2} \left(-\exp \left(-\frac{\mu - \mu_{c3}}{T} \right) \right), \quad (3.4.32)$$

$$\kappa_c = -\frac{1}{2\sqrt{\pi}} T^{-1/2} \text{Li}_{-1/2} \left(-\exp \left(-\frac{\mu - \mu_{c3}}{T} \right) \right), \quad (3.4.33)$$

$$\chi_s = -\frac{1}{4\sqrt{\pi}} T^{-1/2} \text{Li}_{-1/2} \left(-\exp \left(-\frac{\mu - \mu_{c3}}{T} \right) \right). \quad (3.4.34)$$

3.4.4 μ_{c4} : Phase II-IV

The quantum criticality regarding phase IV is complicated due to the two components of Luttinger liquids in this partially polarized phase. In general, in the quantum critical region between phase II and IV, the bound pairs vanish and thus the thermodynamic quantities of unpaired fermions behave as a background, and that of the vanishing one behave as a singular part.

Similar to the analyses of $f_{1/2}$ and $f_{-1/2}$ in the vicinity of μ_{c2} , one can derive the following results in the vicinity of μ_{c4} ,

$$f_{1/2} = \text{Li}_{1/2}(-\zeta) \approx \text{Li}_{1/2} \left(-\exp \left(\frac{2(\mu - \mu_{c4})}{T} \right) \right), \quad (3.4.35)$$

$$f_{-1/2} = \text{Li}_{-1/2}(-\zeta) \approx \text{Li}_{-1/2} \left(-\exp \left(\frac{2(\mu - \mu_{c4})}{T} \right) \right). \quad (3.4.36)$$

In the low-density regime, the cut off of the dressed energy is small, hence $p^b \ll 1$. We rewrite d_1 , γ , and δ by the Sommerfeld expansion

$$\begin{aligned} d_1 &= 2\pi - \int_{-1}^1 dx \frac{\sqrt{1-x^2}}{1 + \exp(x/t)/z} \\ &\approx 2\pi - \int_0^4 dy \frac{1/2}{1 + \exp\left(\frac{y-2-(\mu+2u+B)}{T}\right)} \sqrt{1 - \left(\frac{y-2}{2}\right)^2} \\ &\approx 2\pi - \int_0^\infty dy \frac{\sqrt{y}/2}{1 + \exp\left(\frac{y-2-(\mu+2u+B)}{T}\right)} \\ &\approx 2\pi - \int_0^\infty dy \frac{\sqrt{y}/2}{1 + \exp\left(\frac{y-2-(\mu-\mu_{c4}+\mu_{c4}+2u+B)}{T}\right)} \end{aligned}$$

$$\begin{aligned}
&\approx 2\pi - \int_0^\infty dy \frac{\sqrt{y}/2}{1 + \exp\left(\frac{y-2-(\mu_{c4}+2u+B)}{T}\right)} \\
&\approx 2\pi - \int_0^{\mu_{c4}+2u+B+2} dy \sqrt{y} \\
&= 2\pi - \frac{1}{3} (\mu_{c4} + 2u + B + 2)^{3/2}, \tag{3.4.37}
\end{aligned}$$

and similarly

$$\gamma = \frac{1}{\pi} \int_{-1}^1 dx \frac{1/\sqrt{1-x^2}}{1 + \exp(x/t)/z} \approx \frac{1}{\pi} (\mu_{c4} + 2u + B + 2)^{1/2}, \tag{3.4.38}$$

$$\delta = \frac{1}{\pi} \int_{-1}^1 dx \frac{x/\sqrt{1-x^2}}{1 + \exp(x/t)/z} \approx -\frac{1}{\pi} (\mu_{c4} + 2u + B + 2)^{1/2}. \tag{3.4.39}$$

For simplicity, we denote $q_{c4} = \text{Re}\sqrt{\mu_{c4} + 2u + B + 2}$, and hence obtain

$$d_1 = 2\pi - \frac{1}{3} q_{c4}^3, \quad \gamma = \frac{1}{\pi} q_{c4}, \quad \delta = -\frac{1}{\pi} q_{c4}. \tag{3.4.40}$$

The explicit expression for μ_{c4} is derived from eqs. (2.3.109) and (2.3.110) on the basis of small cut-off in the low-density regime ($|Q| \ll 1$). We approximately recast eq. (2.3.110) into

$$B_{c4} \approx 2\sqrt{1+u^2} - 2 + Q^2 - \frac{2Q^3}{3|u|\pi}, \tag{3.4.41}$$

and thus $Q^2 \approx B + 2 - 2\sqrt{1+u^2}$. We now can write down the approximation of the expression for the critical chemical potential as

$$\mu_{c4} \approx 2|u| - 2\sqrt{1+u^2} + \frac{2}{3|u|\pi} \left(B + 2 - 2\sqrt{1+u^2} \right)^{\frac{3}{2}}, \tag{3.4.42}$$

and then rewrite q_{c4} as

$$\begin{aligned}
q_{c4} &= \sqrt{\mu_{c4} + 2u + B + 2} \\
&\approx \sqrt{B + 2 - 2\sqrt{1+u^2} + \frac{2Q^3}{3\pi|u|}} \\
&= \sqrt{B + 2 - 2\sqrt{1+u^2}} + \frac{1}{3\pi|u|} \left(B + 2 - 2\sqrt{1+u^2} \right). \tag{3.4.43}
\end{aligned}$$

Having made the above preparations, we begin with studying the critical

behavior of the particle density

$$\begin{aligned}
n &= \gamma + (1 - \gamma) \frac{\partial p^b}{\partial \mu} \\
&= \gamma + (1 - \gamma) \frac{-8\tau^{1/2} (2f_{1/2} - \tau f_{3/2})}{\Delta_t} \\
&\approx \gamma - (1 - \gamma) \frac{2}{\sqrt{\pi}} \tau^{1/2} f_{1/2} \\
&= \gamma - (1 - \gamma) \frac{2}{\sqrt{\pi D_1}} T^{1/2} \text{Li}_{1/2} \left(-\exp \left(\frac{2(\mu - \mu_{c4})}{T} \right) \right) \\
&= n_{b4} + \lambda_1 T^{1/2} \text{Li}_{1/2} \left(-\exp \left(\frac{2(\mu - \mu_{c4})}{T} \right) \right), \tag{3.4.44}
\end{aligned}$$

where the background n_{b4} and coefficient λ_1 are given by

$$n_{b4} = \gamma \approx \gamma_0, \tag{3.4.45}$$

$$\lambda_1 = -\frac{2(1 - \gamma)}{\sqrt{\pi D_1}} \approx -\frac{2(1 - \gamma)}{\sqrt{d_1/|u|}} \approx -\frac{2\sqrt{|u|}(1 - q_{c4}/\pi)}{\sqrt{2\pi - q_{c4}^3/3}}. \tag{3.4.46}$$

Similarly, one can derive that

$$\begin{aligned}
m &= \frac{1}{2} \left[\gamma + (1 - \gamma) \frac{\partial p^b}{\partial B} \right] \\
&= \frac{1}{2} \left[\gamma + (1 - \gamma) \frac{-16\delta\tau^{1/2} (f_{1/2} - \tau f_{3/2})}{|u|\Delta_t} \right] \\
&\approx \frac{1}{2} \left[\gamma + (1 - \gamma) \frac{-2\delta}{|u|\sqrt{\pi D_1}} T^{1/2} f_{1/2} \right] \\
&= m_{b4} + \lambda_2 T^{1/2} \text{Li}_{1/2} \left(-\exp \left(\frac{2(\mu - \mu_{c4})}{T} \right) \right), \tag{3.4.47}
\end{aligned}$$

where the background of magnetization m_{b4} and coefficient λ_2 are expressed as

$$m_{b4} = \frac{\gamma}{2} \approx \frac{\gamma_0}{2}, \tag{3.4.48}$$

$$\lambda_2 = -\frac{\delta(1 - \gamma)}{\sqrt{\pi D_1}} \approx \frac{(1 - q_{c4}/\pi)q_{c4}/\pi}{\sqrt{|u|}\sqrt{2\pi - q_{c4}^3/3}}. \tag{3.4.49}$$

We then turn to the compressibility,

$$\begin{aligned}
\kappa_c &= \left(1 - \frac{\partial p^b}{\partial \mu}\right)^2 \gamma' + (1 - \gamma) \frac{\partial^2 p^b}{\partial \mu^2} \\
&= \left[1 - \frac{-8\tau^{1/2}(2f_{1/2} + \tau f_{3/2})}{\Delta_t}\right]^2 \gamma' + (1 - \gamma) \frac{-128f_{-1/2}(16\pi + 7\sqrt{\pi}\tau^{3/2}f_{3/2})}{D_1\tau^{1/2}\Delta_t^3} \\
&\approx \left(1 + \frac{4}{\sqrt{\pi}}\tau^{1/2}f_{1/2} + \frac{6}{\pi}\tau f_{1/2}^2 + \frac{5}{\pi^{3/2}}\tau^{3/2}f_{1/2}^3 - \frac{2}{\sqrt{\pi}}\tau^{3/2}f_{3/2}\right) \gamma'_0 \\
&\quad - (1 - \gamma) \frac{4}{\sqrt{D_1\pi}} T^{-1/2} \text{Li}_{-1/2} \left(-\exp\left(\frac{2(\mu - \mu_{c4})}{T}\right)\right) \\
&= \kappa_{cb4} + \lambda_3 T^{-1/2} \text{Li}_{-1/2} \left(-\exp\left(\frac{2(\mu - \mu_{c4})}{T}\right)\right), \tag{3.4.50}
\end{aligned}$$

where the background compressibility κ_{cb4} and coefficient λ_3 are

$$\kappa_{cb4} = \left(1 + \frac{4}{\sqrt{\pi}}\tau^{1/2}f_{1/2} + \frac{6}{\pi}\tau f_{1/2}^2 + \frac{5}{\pi^{3/2}}\tau^{3/2}f_{1/2}^3 - \frac{2}{\sqrt{\pi}}\tau^{3/2}f_{3/2}\right) \gamma'_0, \tag{3.4.51}$$

$$\lambda_3 = -\frac{4(1 - \gamma)}{\sqrt{d_1/|u|}} \approx -\frac{4\sqrt{|u|}(1 - q_{c4}/\pi)}{\sqrt{2\pi - q_{c4}^3/3}}. \tag{3.4.52}$$

The susceptibility is

$$\begin{aligned}
\chi_s &= \frac{1}{2} \left(1 - \frac{\partial p^b}{\partial B}\right)^2 \gamma' + \frac{1}{2} (1 - \gamma) \frac{\partial^2 p^b}{\partial B^2}, \\
&= \frac{1}{2} \left[1 - \frac{-16\delta\tau^{1/2}(f_{1/2} - \tau f_{3/2})}{|u|\Delta_t}\right]^2 \gamma' + \frac{1}{2} (1 - \gamma) \left[-\frac{16\delta'\tau^{1/2}(f_{1/2} - \tau f_{3/2})}{|u|\Delta_t}\right. \\
&\quad \left.- \frac{4096\pi\delta^2 f_{-1/2} \left(16\sqrt{\pi}f_{1/2} - 8\tau^{1/2}f_{1/2}^2 - 5\tau^{3/2}f_{1/2}f_{3/2} - 4\sqrt{\pi}\tau f_{3/2}\right)}{u^2 D_1 \tau^{1/2} (4f_{1/2} - \tau f_{3/2}) \Delta_t^4}\right] \\
&\approx \frac{1}{2} \left[1 - \frac{-16\delta\tau^{1/2}(f_{1/2} - \tau f_{3/2})}{|u|\Delta_t}\right]^2 \gamma' - \frac{1}{2} (1 - \gamma) \frac{16\delta'\tau^{1/2}(f_{1/2} - \tau f_{3/2})}{|u|\Delta_t} \\
&\quad - \frac{1}{2} (1 - \gamma) \frac{4096\pi\delta^2 16\sqrt{\pi}f_{1/2}}{u^2 \sqrt{D_1} 4f_{1/2} (8\sqrt{\pi})^4} T^{-1/2} f_{-1/2} \\
&\approx \chi_{sb4} + \lambda_4 T^{-1/2} \text{Li}_{-1/2} \left(-\exp\left(\frac{2(\mu - \mu_{c4})}{T}\right)\right), \tag{3.4.53}
\end{aligned}$$

where the background for the susceptibility χ_{sb4} and coefficient λ_4 read

$$\begin{aligned}
\chi_{sb4} &= \frac{1}{2} \left[1 - \frac{-16\delta\tau^{1/2}(f_{1/2} - \tau f_{3/2})}{|u|\Delta_t} \right]^2 \gamma' - \frac{1}{2}(1-\gamma) \frac{16\delta'\tau^{1/2}(f_{1/2} - \tau f_{3/2})}{|u|\Delta_t} \\
&\approx \frac{1}{2} \gamma' \left\{ 1 + \frac{32\delta\tau^{1/2}(f_{1/2} - \tau f_{3/2})}{|u|8\sqrt{\pi} \left[1 - \left(\frac{1}{2\sqrt{\pi}}\tau^{1/2}f_{1/2} - \frac{1}{8\sqrt{\pi}}\tau^{3/2}f_{3/2} \right) \right]} \right\} - \frac{1}{2} \frac{16\delta'\tau^{1/2}(f_{1/2} - \tau f_{3/2})}{|u|8\sqrt{\pi} \left[1 - \left(\frac{1}{2\sqrt{\pi}}\tau^{1/2}f_{1/2} - \frac{1}{8\sqrt{\pi}}\tau^{3/2}f_{3/2} \right) \right]} \\
&\approx \frac{1}{2} \gamma' \left[1 + \frac{4\delta\tau^{1/2}(f_{1/2} - \tau f_{3/2})}{|u|\sqrt{\pi}} \left(1 + \frac{1}{2\sqrt{\pi}}\tau^{1/2}f_{1/2} + \frac{1}{4\pi}\tau f_{1/2}^2 \right) \right] \\
&\quad - \frac{1}{2}(1-\gamma) \frac{2\delta'\tau^{1/2}(f_{1/2} - \tau f_{3/2})}{|u|\sqrt{\pi}} \left(1 + \frac{1}{2\sqrt{\pi}}\tau^{1/2}f_{1/2} + \frac{1}{4\pi}\tau f_{1/2}^2 \right) \\
&= \frac{1}{2} \gamma'_0 + \frac{2\delta_0\gamma'_0 - (1-\gamma_0)\delta'_0}{|u|\sqrt{\pi}} \left(\tau^{1/2}f_{1/2} + \frac{1}{2\sqrt{\pi}}\tau f_{1/2}^2 + \frac{1}{4\pi}\tau^{3/2}f_{1/2}^3 - \tau^{3/2}f_{3/2} \right)
\end{aligned} \tag{3.4.54}$$

$$\lambda_4 = -\frac{1}{2}(1-\gamma) \frac{4096\pi\delta^2 16\sqrt{\pi}f_{1/2}}{u^2\sqrt{D_1}4f_{1/2}(8\sqrt{\pi})^4} \approx -\frac{2\delta^2(1-\gamma)}{\sqrt{|u|^3d_1}} \approx -\frac{2(q_{c4}/\pi)^2(1-q_{c4}/\pi)}{\sqrt{|u|^3}\sqrt{2\pi-q_{c4}^3/3}}. \tag{3.4.55}$$

The critical behaviour of the thermodynamic quantities between phases II and IV is

$$n = n_{b4} + \lambda_1 T^{1/2} \text{Li}_{1/2} \left(-\exp \left(\frac{2(\mu - \mu_{c4})}{T} \right) \right), \tag{3.4.56}$$

$$m = m_{b4} + \lambda_2 T^{1/2} \text{Li}_{1/2} \left(-\exp \left(\frac{2(\mu - \mu_{c4})}{T} \right) \right), \tag{3.4.57}$$

$$\kappa_c = \kappa_{cb4} + \lambda_3 T^{-1/2} \text{Li}_{-1/2} \left(-\exp \left(\frac{2(\mu - \mu_{c4})}{T} \right) \right), \tag{3.4.58}$$

$$\chi_s = \chi_{sb4} + \lambda_4 T^{-1/2} \text{Li}_{-1/2} \left(-\exp \left(\frac{2(\mu - \mu_{c4})}{T} \right) \right), \tag{3.4.59}$$

where

$$n_{b4} = \gamma_0, \quad \lambda_1 = -\frac{2\sqrt{|u|}(1-q_{c4}/\pi)}{\sqrt{2\pi-q_{c4}^3/3}}, \tag{3.4.60}$$

$$m_{b4} = \frac{\gamma_0}{2}, \quad \lambda_2 = \frac{(1-q_{c4}/\pi)q_{c4}/\pi}{\sqrt{|u|}\sqrt{2\pi-q_{c4}^3/3}}, \tag{3.4.61}$$

$$\kappa_{cb4} = \left(1 + \frac{4}{\sqrt{\pi}}\tau^{1/2}f_{1/2} + \frac{6}{\pi}\tau f_{1/2}^2 + \frac{5}{\pi^{3/2}}\tau^{3/2}f_{1/2}^3 - \frac{2}{\sqrt{\pi}}\tau^{3/2}f_{3/2} \right) \gamma'_0,$$

$$\lambda_3 = -\frac{4\sqrt{|u|}(1-q_{c4}/\pi)}{\sqrt{2\pi-q_{c4}^3/3}}, \quad (3.4.62)$$

$$\begin{aligned} \chi_{sb4} &= \frac{1}{2}\gamma'_0 + \frac{2\delta_0\gamma'_0 - (1-\gamma_0)\delta'_0}{|u|\sqrt{\pi}} \left(\tau^{1/2}f_{1/2} + \frac{1}{2\sqrt{\pi}}\tau f_{1/2}^2 + \frac{1}{4\pi}\tau^{3/2}f_{1/2}^3 - \tau^{3/2}f_{3/2} \right), \\ \lambda_4 &= -\frac{2(q_{c4}/\pi)^2(1-q_{c4}/\pi)}{\sqrt{|u|^3}\sqrt{2\pi-q_{c4}^3/3}}, \end{aligned} \quad (3.4.63)$$

with

$$q_{c4} = \sqrt{B+2-2\sqrt{1+u^2}} + \frac{1}{3\pi|u|} \left(B+2-2\sqrt{1+u^2} \right). \quad (3.4.64)$$

3.4.5 μ_{c5} : Phase IV-V

In the vicinity of μ_{c5} , the vanishing component in the FFLO-like phase IV is the unpaired fermion. In contrast to the situation for the quantum critical region between phases II and IV, here the thermodynamic quantities of bound pairs play a role of background, and those of unpaired fermions are responsible for the singular part.

As before we start from the particle density,

$$\begin{aligned} n &= \frac{\partial p^b}{\partial \mu} + \left(1 - \frac{\partial p^b}{\partial \mu}\right)\gamma \\ &\approx \frac{\partial p^b}{\partial \mu} + \left(1 - \frac{\partial p^b}{\partial \mu}\right) \frac{-T^{1/2}}{2\sqrt{\pi}} \text{Li}_{1/2} \left(-\exp \left(\frac{\mu - \mu_{c5}}{T} \right) \right) \\ &= n_{b5} + \lambda_5 T^{1/2} \text{Li}_{1/2} \left(-\exp \left(\frac{\mu - \mu_{c5}}{T} \right) \right), \end{aligned} \quad (3.4.65)$$

where the transformation from γ to the polylogarithm is similar to the analysis for the quantum criticality between phase I and II, and

$$\begin{aligned} n_{b5} &= \frac{\partial p^b}{\partial \mu} = -\frac{8\tau^{1/2}(2f_{1/2} - \tau f_{3/2})}{\Delta_t} \\ &\approx -\frac{1}{\sqrt{\pi}} \left(2\tau^{1/2}f_{1/2} - \tau^{3/2}f_{3/2} \right) \left(1 + \frac{1}{2\sqrt{\pi}}\tau^{1/2}f_{1/2} + \frac{1}{4\pi}\tau f_{1/2}^2 \right), \\ &\approx -\frac{2}{\sqrt{\pi}}\tau^{1/2}f_{1/2} - \frac{1}{\pi}\tau f_{1/2}^2 - \frac{1}{2\pi^{3/2}}\tau^{3/2}f_{1/2}^3 + \frac{1}{\sqrt{\pi}}\tau^{3/2}f_{3/2}, \\ \lambda_5 &= -\frac{1}{2\sqrt{\pi}} \left(1 - \frac{\partial p^b}{\partial \mu} \right) \approx -\frac{1}{2\sqrt{\pi}} \left(1 + \frac{2}{\sqrt{\pi}}\tau^{1/2}f_{1/2} \right) \end{aligned} \quad (3.4.66)$$

$$\begin{aligned}
&\approx -\frac{1}{2\sqrt{\pi}}(1+2\sqrt{\frac{|u|}{d_0}}T^{1/2}f_{1/2}) \\
&\approx -\frac{1}{2\sqrt{\pi}}(1+2\sqrt{\frac{|u|}{d_0}}T^{1/2}g_{1/2}).
\end{aligned} \tag{3.4.67}$$

To eliminate the influence of temperature in λ_5 , we begin with the Sommerfeld expansion for d_0 , δ_0 and γ_0 , for which the results are similar to those in the vicinity of μ_{c4} ,

$$d_0 = 2\pi - \frac{1}{3}q_{c5}^3, \quad \gamma = \frac{1}{\pi}q_{c5}, \quad \delta = -\frac{1}{\pi}q_{c5}, \tag{3.4.68}$$

with

$$q_{c5} = \text{Re}\sqrt{\mu_{c5} + 2u + B + 2}. \tag{3.4.69}$$

We then consider $T^{1/2}g_{1/2}$, and by the Sommerfeld expansion we derive

$$\begin{aligned}
T^s \cdot g_s &= T^s \cdot \text{Li}_s(-\zeta) = T^s \frac{-1}{\Gamma(s)} \int_0^\infty dx \frac{x^{s-1}}{1 + e^{x/\zeta_0}} \\
&= \frac{-1}{\Gamma(s)} \int_0^\infty dy \frac{y^{s-1}}{1 + \exp\left(\frac{y - (2\mu + \frac{d_0}{|u|\pi})}{T}\right)} \\
&\approx \frac{-1}{\Gamma(s)} \int_0^\infty dy \frac{y^{s-1}}{1 + \exp\left(\frac{y - (2\mu_{c5} + \frac{d_0}{|u|\pi})}{T}\right)} \\
&= \frac{-1}{\Gamma(s+1)} \left(2\mu_{c5} + \frac{d_0}{|u|\pi}\right)^s \\
&\approx \frac{-1}{\Gamma(s+1)} \left(2\mu_{c5} + \frac{2}{|u|} - \frac{q_{c5}^3}{3|u|\pi}\right)^s.
\end{aligned} \tag{3.4.70}$$

Hence, using above result for $T^s g_s$, we can write down the particle density in this critical region as

$$n = n_{b5} + \lambda_5 T^{1/2} \text{Li}_{1/2}\left(-\exp\left(\frac{\mu - \mu_{c5}}{T}\right)\right), \tag{3.4.71}$$

where

$$n_{b5} = -\frac{2}{\sqrt{\pi}}\tau^{1/2}f_{1/2} - \frac{1}{\pi}\tau f_{1/2}^2 - \frac{1}{2\pi^{3/2}}\tau^{3/2}f_{1/2}^3 + \frac{1}{\sqrt{\pi}}\tau^{3/2}f_{3/2}, \tag{3.4.72}$$

$$\lambda_5 = -\frac{1}{2\sqrt{\pi}} \left(1 - \frac{4}{\pi} \sqrt{1 + \frac{2\pi|u|\mu_{c5}}{2\pi - q_{c5}^3/3}} \right). \quad (3.4.73)$$

Obviously, here we need an alternative expression for μ_{c5} . Remember that under the condition $A \ll 1$, the PB(IV,V) is approximated as

$$\mu_{c5} \approx 2|u| - B - 2 + \frac{8\sqrt{2}}{3\pi|u|\alpha} \left[2\sqrt{u^2 + 1} - B - 2 \right]^{\frac{3}{2}}. \quad (3.4.74)$$

The magnetization is expressed by

$$\begin{aligned} m &= \frac{1}{2} \left[\frac{\partial p^b}{\partial B} + \left(1 - \frac{\partial p^b}{\partial B} \right) \gamma \right] \\ &\approx -\frac{1}{4\sqrt{\pi}} T^{1/2} \text{Li}_{1/2} \left(-\exp \left(\frac{\mu - \mu_{c5}}{T} \right) \right), \end{aligned} \quad (3.4.75)$$

where the transformation from γ to the polylogarithm is the same as that in the vicinity of μ_{c1} .

Now we study the critical behaviour of κ_c , for which

$$\begin{aligned} \kappa_c &= \left(1 - \frac{\partial p^b}{\partial \mu} \right)^2 \gamma' + (1 - \gamma) \frac{\partial^2 p^b}{\partial \mu^2} \\ &\approx (1 - \gamma_0) \frac{\partial^2 p^b}{\partial \mu^2} + \left(1 - \frac{\partial p^b}{\partial \mu} \right)^2 \frac{-1}{2\sqrt{\pi}} T^{-1/2} \text{Li}_{-1/2} \left(-\exp \left(\frac{\mu - \mu_{c5}}{T} \right) \right) \\ &= \kappa_{cb5} + \lambda_6 T^{-1/2} \text{Li}_{-1/2} \left(-\exp \left(\frac{\mu - \mu_{c5}}{T} \right) \right), \end{aligned} \quad (3.4.76)$$

where the transformation from γ' to the polylogarithm is similar to the analysis for the quantum criticality between phase I and II, and

$$\begin{aligned} \kappa_{cb5} &= (1 - \gamma_0) \frac{\partial^2 p^b}{\partial \mu^2} = (1 - \gamma_0) \frac{-128 f_{-1/2} \left(16\pi - \sqrt{\pi} \tau^{3/2} f_{3/2} \right)}{D_1 \tau^{1/2} \Delta_t^3} \\ &\approx - (1 - \gamma_0) \frac{f_{-1/2}}{D_1 \tau^{1/2}} \frac{\left(\frac{4}{\sqrt{\pi}} - \frac{1}{4\pi} \tau^{3/2} f_{3/2} \right)}{\left[1 - \left(\frac{1}{2\sqrt{\pi}} \tau^{1/2} f_{1/2} - \frac{1}{8\sqrt{\pi}} \tau^{3/2} f_{3/2} \right) \right]^3} \\ &\approx - (1 - \gamma_0) \frac{f_{-1/2}}{D_1 \tau^{1/2}} \left(\frac{4}{\sqrt{\pi}} - \frac{1}{4\pi} \tau^{3/2} f_{3/2} \right) \left[1 + 3 \left(\frac{1}{2\sqrt{\pi}} \tau^{1/2} f_{1/2} - \frac{1}{8\sqrt{\pi}} \tau^{3/2} f_{3/2} \right) \right. \\ &\quad \left. + 6 \left(\frac{1}{2\sqrt{\pi}} \tau^{1/2} f_{1/2} \right)^2 + 10 \left(\frac{1}{2\sqrt{\pi}} \tau^{1/2} f_{1/2} \right)^3 \right] \end{aligned}$$

$$\approx -(1-\gamma_0) \frac{f_{-1/2}}{D_1 \tau^{1/2}} \left(\frac{4}{\sqrt{\pi}} + \frac{6}{\pi} \tau^{1/2} f_{1/2} + \frac{6}{\pi^{3/2}} \tau f_{1/2}^2 + \frac{5}{\pi^2} \tau^{3/2} f_{1/2}^3 - \frac{7}{4\pi} \tau^{3/2} f_{3/2} \right), \quad (3.4.77)$$

$$\begin{aligned} \lambda_6 &= -\frac{1}{2\sqrt{\pi}} \left(1 - \frac{\partial p^b}{\partial \mu} \right)^2 = -\frac{1}{2\sqrt{\pi}} \left(1 - \frac{-8\tau^{1/2} (2f_{1/2} - \tau f_{3/2})}{\Delta_t} \right)^2 \\ &\approx -\frac{1}{2\sqrt{\pi}} \left(1 + 4\sqrt{\frac{|u|}{d_0}} T^{1/2} g_{1/2} \right) \\ &\approx -\frac{1}{2\sqrt{\pi}} \left(1 - \frac{8}{\pi} \sqrt{1 + \frac{2\pi|u|\mu_{c5}}{2\pi - q_{c5}^3/3}} \right). \end{aligned} \quad (3.4.78)$$

The case of susceptibility is rather simple, with

$$\begin{aligned} \chi_s &= \frac{1}{2} \left[\left(1 - \frac{\partial p^b}{\partial B} \right)^2 \gamma' + (1-\gamma) \frac{\partial^2 p^b}{\partial B^2} \right] \approx \frac{1}{2} \gamma' \\ &= -\frac{1}{4\sqrt{\pi}} T^{-1/2} \text{Li}_{-1/2} \left(-\exp \left(\frac{\mu - \mu_{c5}}{T} \right) \right), \end{aligned} \quad (3.4.79)$$

The critical behaviour of the thermodynamic quantities between phases IV and V is thus

$$n = n_{b5} + \lambda_5 T^{1/2} \text{Li}_{1/2} \left(-\exp \left(\frac{\mu - \mu_{c5}}{T} \right) \right), \quad (3.4.80)$$

$$m = -\frac{1}{4\sqrt{\pi}} T^{1/2} \text{Li}_{1/2} \left(-\exp \left(\frac{\mu - \mu_{c5}}{T} \right) \right), \quad (3.4.81)$$

$$\kappa_c = \kappa_{cb5} + \lambda_6 T^{-1/2} \text{Li}_{-1/2} \left(-\exp \left(\frac{\mu - \mu_{c5}}{T} \right) \right), \quad (3.4.82)$$

$$\chi_s = -\frac{1}{4\sqrt{\pi}} T^{-1/2} \text{Li}_{-1/2} \left(-\exp \left(\frac{\mu - \mu_{c5}}{T} \right) \right), \quad (3.4.83)$$

where

$$\begin{aligned} n_{b5} &= -\frac{2}{\sqrt{\pi}} \tau^{1/2} f_{1/2} - \frac{1}{\pi} \tau f_{1/2}^2 - \frac{1}{2\pi^{3/2}} \tau^{3/2} f_{1/2}^3 + \frac{1}{\sqrt{\pi}} \tau^{3/2} f_{3/2}, \\ \lambda_5 &= -\frac{1}{2\sqrt{\pi}} \left(1 - \frac{4}{\pi} \sqrt{1 + \frac{2\pi|u|\mu_{c5}}{2\pi - q_{c5}^3/3}} \right); \\ \kappa_{cb5} &= -(1-\gamma_0) \frac{f_{-1/2}}{D_1 \tau^{1/2}} \left(\frac{4}{\sqrt{\pi}} + \frac{6}{\pi} \tau^{1/2} f_{1/2} + \frac{6}{\pi^{3/2}} \tau f_{1/2}^2 + \frac{5}{\pi^2} \tau^{3/2} f_{1/2}^3 - \frac{7}{4\pi} \tau^{3/2} f_{3/2} \right), \end{aligned} \quad (3.4.84)$$

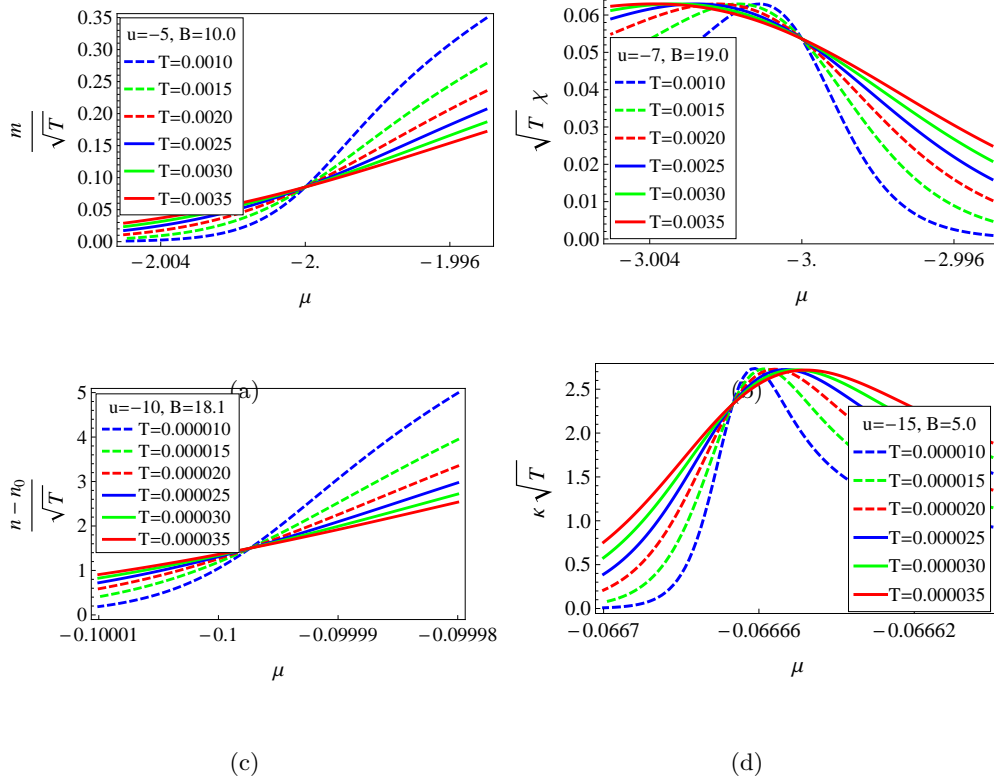


Figure 3.3: Scaling laws for thermodynamic quantities vs chemical potential at different temperatures. The intersection points in (a), (b), (c) and (d) give the critical points for phase transitions (I-II), (II-III), (II-IV) and (I-V), respectively.

$$\lambda_6 = -\frac{1}{2\sqrt{\pi}} \left(1 - \frac{8}{\pi} \sqrt{1 + \frac{2\pi|u|\mu_{c5}}{2\pi - q_{c5}^3/3}} \right), \quad (3.4.85)$$

with

$$\mu_{c5} = 2|u| - B - 2 + \frac{8\sqrt{2}}{3\pi|u|\alpha} \left[2\sqrt{u^2 + 1} - B - 2 \right]^{\frac{3}{2}}, \quad (3.4.86)$$

$$q_{c5} = \text{Re} \sqrt{\mu_{c5} + 2u + B + 2}. \quad (3.4.87)$$

3.4.6 Summary for Quantum Criticality

The five series of scaling forms can be cast into a universal scaling form [92–95], if we let n and κ stand for the first and second order thermodynamic quantities respectively. We thus have

$$n(\mu, B, T) = n_0(\mu, B, T) + T^{d/z+1-(1/\nu z)} \mathcal{G} \left(\frac{\mu - \mu_c}{T^{1/\nu z}} \right),$$

$$\kappa(\mu, B, T) = \kappa_0(\mu, B, T) + T^{d/z+1-(2/vz)} \mathcal{F}\left(\frac{\mu - \mu_c}{T^{1/vz}}\right), \quad (3.4.88)$$

where n_0 and κ_0 are the regular parts, with $\mathcal{G}(x) = \text{Li}_{\frac{1}{2}}(x)$, $\mathcal{F}(x) = \text{Li}_{-\frac{1}{2}}(x)$ the scaling functions. From the above scaling forms, one can read off the dynamical exponent $z = 2$ and correlation critical exponent $\nu = 1/2$. We demonstrate these scaling relations in fig. 3.3.

The above scaling forms are observed to give the same critical exponents which characterize the universality class of free-fermion criticality. An intuitive explanation for this result is that the phase transitions in the 1D Hubbard model have the common feature of one branch of Fermi sea vanishing at the critical point. This naturally leads to a change in dispersion, i.e., a linear dispersion vanishes while a quadratic dispersion is created when the phase transition occurs. This change in dispersion underlies a universality class of quantum criticality [96, 97].

Last but not least, we would like to make a remark for the phase transition between phase II and III. In contrast to the 1D attractive $SU(2)$ Fermi gas, here the half-filling phase of the 1D attractive Hubbard model contributes a constant regular part to the thermodynamic quantities due to its unique band-filling. This is the only phase transition in this model which takes place because of band-filling but not vanishing of one branch of Fermi sea.

Free-Fermi Liquids and Additivity Rules in FFLO-like Phase

Fermi liquid theory is believed to break down in 1D strongly correlated systems due to the absence of well defined quasi-particles. Consequently the TLL theory is generally believed to describe the collective low-lying excitations in 1D many-body systems. Despite such a big difference in the microscopic origins of the two low-energy theories, both the Fermi liquid and the TLL share a common feature – a small distortion of the Fermi surface or Fermi points results in the universal low-energy physics of many-body systems. From the results of Chapter 3, we have observed that at very low temperatures the low-energy physics of the FFLO-like state is governed by the universality class of a two-component TLL.

However, in view of the macroscopic properties of the 1D attractive Hubbard model, we argue that such a universality class of two-component TLL reveals an important free-Fermi liquid nature. In order to show this rather elegant nature, we will introduce two effective chemical potentials for the excess fermions and bound pairs on a 1D lattice. Then we will show that the thermodynamic properties in the FFLO-like phase behave like two independent free-Fermi liquids. In particular, we find simple additivity rules for the compressibility and susceptibility which represent a universal characteristic of quantum liquids at the renormalization fixed point.

To begin with, we would like to make some preparations for later discussion of the free-Fermi liquids. The zero temperature TBA equations (2.3.96) and (2.3.97) in the low density regime can be approximated as

$$\epsilon^u(k) = k^2 - \mu^u - a_1 \star \epsilon^b(k), \quad (4.0.1)$$

$$\epsilon^b(\Lambda) = \alpha_1 \Lambda^2 - \alpha_1 \mu^b - a_1 \star \epsilon^u(\Lambda) - a_2 \star \epsilon^b(\Lambda), \quad (4.0.2)$$

where $a_m \star \epsilon^\gamma(x) = \int_{-Q_\gamma}^{Q_\gamma} dy a_m(x-y) \epsilon^\gamma(y)$ with Q_γ ($\gamma = u, b$) being the Fermi point of $\epsilon^\gamma(y)$. In the above equations, we have introduced two effective chemical po-

tentials for unpaired fermions and bound pairs as

$$\mu^u = \mu + B + 2u + 2, \quad (4.0.3)$$

$$\mu^b = \frac{2}{\alpha_1} \left(\mu + 2\sqrt{u^2 + 1} - 2|u| \right). \quad (4.0.4)$$

The effective chemical potential of the bound pairs reveals a deep physical insight into the crossover from Bose-Einstein condensate to Bardeen-Cooper-Schrieffer superconductor. Later we shall see these effective chemical potentials reveal an important free quantum liquid nature.

This new form of TBA equations is useful to access the ground state properties, such as sound velocities, stiffness and effective chemical potentials. By virtue of eqs. (4.0.1) and (4.0.2), we can rewrite the free energy per site eq. (2.3.44) as

$$f = u + \int_{-Q_u}^{Q_u} \frac{dk}{2\pi} \epsilon^u(k) + \int_{-Q_b}^{Q_b} \frac{d\Lambda}{2\pi} \beta_1 \epsilon^b(\Lambda), \quad (4.0.5)$$

where $\kappa(k) = \epsilon^u(k)$ and $\epsilon'_u(\Lambda) = \epsilon^b(\Lambda)$.

Here we introduce two parameters α_1 and β_1 ¹, which are functions of u and represent the lattice effect

$$\alpha_1 = \int_{-\pi}^{\pi} dk \frac{2|u| \cos^2 k (u^2 - 3 \sin^2 k)}{\pi (\sin^2 k + u^2)^3}, \quad (4.0.6)$$

$$\beta_1 = \int_{-\pi}^{\pi} dk a_1(\sin k). \quad (4.0.7)$$

4.1 Luttinger Parameters

In this section, we figure out the TLL parameters of the 1D attractive Hubbard model and compare them with those of the 1D attractive $SU(2)$ Fermi gas [91]. The basic idea is to express the effective chemical potentials in terms of the Fermi points by employing iteration of the eqs. (4.0.1) and (4.0.2). Using the fact that the two dressed energies vanish at their corresponding Fermi points, we express those Fermi points in terms of the densities of unpaired fermions and bound pairs. We shall see that this process leads to a separation of two free-Fermi liquids in the ground state energy per site.

For convenience, we rescale the TBA equations (4.0.1) and (4.0.2) by defining

¹ α_1 has appeared in eq. (2.3.114), the phase boundary between phase IV and V.

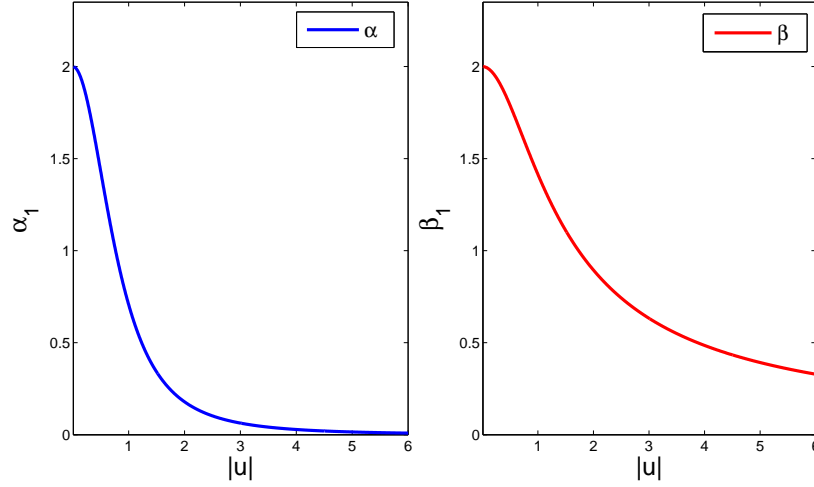


Figure 4.1: The lattice interacting parameters for the length-1 $k-\Lambda$ strings as a function of the interaction strength u . The parameter α_1 strongly affects the band dispersion of bound pairs. The parameter β_1 presents a lattice contribution to the free energy of the pairs.

$\tilde{\epsilon}^\gamma = \epsilon^\gamma/u^2$, $\tilde{\mu}^\gamma = \mu^\gamma/u^2$, $\tilde{Q}_\gamma = Q_\gamma/|u|$, and $\bar{a}_n(x) = \frac{n}{\pi} \frac{1}{n^2+x^2}$, with result

$$\tilde{\epsilon}^u(\tilde{k}) = \tilde{k}^2 - \tilde{\mu}^u - \bar{a}_1 \star \tilde{\epsilon}^b(\tilde{k}), \quad (4.1.1)$$

$$\tilde{\epsilon}^b(\tilde{\Lambda}) = \alpha_1 \tilde{\Lambda}^2 - \alpha_1 \tilde{\mu}^b - \bar{a}_1 \star \tilde{\epsilon}^u(\tilde{\Lambda}) - \bar{a}_2 \star \tilde{\epsilon}^b(\tilde{\Lambda}). \quad (4.1.2)$$

In order to simplify the lengthy iterations, we introduce a vector presentation for the above rescaled TBA equations. In view of the properties of even functions, we utilize the base $\{\tilde{k}^{2n}\}$ and $\{\tilde{\Lambda}^{2n}\}$ ($n = 0, 1, 2, \dots$) to expand these scalar equations, thus we have

$$\vec{\epsilon}^u = \vec{V}^u - \mathbf{A}^1(\tilde{Q}_b) \vec{\epsilon}^b, \quad (4.1.3)$$

$$\vec{\epsilon}^b = \vec{V}^b - \mathbf{A}^1(\tilde{Q}_u) \vec{\epsilon}^u - \mathbf{A}^2(\tilde{Q}_b) \vec{\epsilon}^b. \quad (4.1.4)$$

The vectors $\vec{V}^u = [-\tilde{\mu}^u, 1, 0, \dots]^{Tr}$ and $\vec{V}^b = [-\alpha_1 \tilde{\mu}^b, \alpha_1, 0, \dots]^{Tr}$ are the driving terms and the superscript Tr represents transpose operation. The matrix $\mathbf{A}^n(\tilde{Q}_\gamma) \vec{\epsilon}$ corresponds to the integral $\int_{-\tilde{Q}_\gamma}^{\tilde{Q}_\gamma} dy \bar{a}_n(x-y) \tilde{\epsilon}^\gamma(y)$, and in general its element can be explicitly expressed by

$$\{\mathbf{A}^n(\tilde{Q}_\gamma)\}_{jl} = \frac{2}{\pi} \sum_{0 \leq j \leq i < \infty} \frac{(-1)^i C_{2i}^{2j} \tilde{Q}_\gamma^{2i-2j+2l+1}}{n^{2i+1} (2i-2j+2l+1)}, \quad (4.1.5)$$

with $j, l = 0, 1, 2, \dots$

Furthermore, as we only retain the first few leading terms, both $\vec{\epsilon}^n$ and $\mathbf{A}^n(\tilde{Q}_\gamma)$ can be expanded as sums of a few leading orders with respect to \tilde{Q}_γ ,

$$\vec{\epsilon}^\gamma = \vec{\epsilon}_{(0)}^\gamma + \vec{\epsilon}_{(1)}^\gamma + \vec{\epsilon}_{(2)}^\gamma + \vec{\epsilon}_{(3)}^\gamma + \dots \quad (4.1.6)$$

$$\mathbf{A}^n(\tilde{Q}_\gamma) = \mathbf{A}_{(1)}^n(\tilde{Q}_\gamma) + \mathbf{A}_{(3)}^n(\tilde{Q}_\gamma) + \mathbf{A}_{(5)}^n(\tilde{Q}_\gamma) + \dots \quad (4.1.7)$$

The explicit expressions of the first two orders of $\mathbf{A}_{(j)}^n(\tilde{Q}_\gamma)$ are

$$\mathbf{A}_{(1)}^1(\tilde{Q}_\gamma) = \frac{2}{\pi} \begin{bmatrix} 1 & 0 & \dots \\ -1 & 0 & \dots \\ \vdots & & \end{bmatrix} \tilde{Q}_\gamma, \quad \mathbf{A}_{(1)}^2(\tilde{Q}_\gamma) = \frac{1}{\pi} \begin{bmatrix} 1 & 0 & \dots \\ -\frac{1}{4} & 0 & \dots \\ \vdots & & \end{bmatrix} \tilde{Q}_\gamma, \quad (4.1.8)$$

$$\mathbf{A}_{(3)}^1(\tilde{Q}_\gamma) = \frac{2}{\pi} \begin{bmatrix} -\frac{1}{3} & \frac{1}{3} & \dots \\ 2 & \frac{1}{3} & \dots \\ \vdots & & \end{bmatrix} \tilde{Q}_\gamma^3, \quad \mathbf{A}_{(3)}^2(\tilde{Q}_\gamma) = \frac{1}{\pi} \begin{bmatrix} -\frac{1}{12} & \frac{1}{3} & \dots \\ \frac{1}{8} & -\frac{1}{12} & \dots \\ \vdots & & \end{bmatrix} \tilde{Q}_\gamma^3. \quad (4.1.9)$$

Substituting expansions of eqs. (4.1.6) and (4.1.7) into the TBA equations of vectorial form in eqs. (4.1.3) and (4.1.4), and sorting terms order by order, leads to a set of recurrence equations,

$$\begin{aligned} \vec{\epsilon}_{(0)}^u &= \vec{V}^u, & \vec{\epsilon}_{(0)}^b &= \vec{V}^b, \\ \vec{\epsilon}_{(1)}^u &= -\mathbf{A}_{(1)}^1(\tilde{Q}_b) \vec{\epsilon}_{(0)}^b, \\ \vec{\epsilon}_{(1)}^b &= -\mathbf{A}_{(1)}^1(\tilde{Q}_u) \vec{\epsilon}_{(0)}^u - \mathbf{A}_{(1)}^2(\tilde{Q}_b) \vec{\epsilon}_{(0)}^b, \\ \vec{\epsilon}_{(2)}^u &= -\mathbf{A}_{(1)}^1(\tilde{Q}_b) \vec{\epsilon}_{(1)}^b, \\ \vec{\epsilon}_{(2)}^b &= -\mathbf{A}_{(1)}^1(\tilde{Q}_u) \vec{\epsilon}_{(1)}^u - \mathbf{A}_{(1)}^2(\tilde{Q}_b) \vec{\epsilon}_{(1)}^b, \\ \vec{\epsilon}_{(3)}^u &= -\mathbf{A}_{(1)}^1(\tilde{Q}_b) \vec{\epsilon}_{(2)}^b - \mathbf{A}_{(3)}^1(\tilde{Q}_b) \vec{\epsilon}_{(0)}^b, \\ \vec{\epsilon}_{(3)}^b &= -\mathbf{A}_{(1)}^1(\tilde{Q}_u) \vec{\epsilon}_{(2)}^u - \mathbf{A}_{(1)}^2(\tilde{Q}_b) \vec{\epsilon}_{(2)}^b - \mathbf{A}_{(3)}^1(\tilde{Q}_u) \vec{\epsilon}_{(0)}^u - \mathbf{A}_{(3)}^2(\tilde{Q}_b) \vec{\epsilon}_{(0)}^b, \end{aligned}$$

whose solutions give rise to

$$\tilde{\epsilon}^u(k) = k^2 - \mu^u + \frac{2\alpha_1}{\pi} \mu^b \tilde{Q}_b - \frac{2\alpha_1}{3\pi} \tilde{Q}_b^3, \quad (4.1.10)$$

$$\tilde{\epsilon}^b(\Lambda) = \alpha_1 \Lambda^2 - \alpha_1 \mu^b + \frac{2}{\pi} \mu^u \tilde{Q}_u + \frac{\alpha_1}{\pi} \mu^b \tilde{Q}_b - \frac{2}{3\pi} \tilde{Q}_u^3 - \frac{\alpha_1}{3\pi} \tilde{Q}_b^3. \quad (4.1.11)$$

Let $\tilde{\epsilon}^\gamma(\tilde{Q}_\gamma) = 0$, then one can readily express $\tilde{\mu}^\gamma$ in terms of \tilde{Q}_η ($\gamma, \eta = u, b$),

which can be rewritten in a vector equation

$$\begin{bmatrix} \tilde{\mu}^u \\ \alpha_1 \tilde{\mu}^b \end{bmatrix} = \left(\mathbf{I} + \frac{2}{3} \mathbf{T} \right) \begin{bmatrix} \tilde{Q}_u^2 \\ \alpha_1 \tilde{Q}_b^2 \end{bmatrix}, \quad (4.1.12)$$

where the matrix \mathbf{T} is given by

$$\mathbf{T} = \frac{1}{\pi} \begin{bmatrix} 0 & 2\tilde{Q}_b \\ 2\tilde{Q}_u & \tilde{Q}_b \end{bmatrix}. \quad (4.1.13)$$

Finally, the free energy per site eq. (4.0.5) is expressed in terms of the Fermi points Q_u and Q_b , with result

$$f = -\frac{2}{3\pi} (Q_u^3 + \alpha_1 \beta_1 Q_b^3) + u. \quad (4.1.14)$$

We now proceed to obtain the particle densities in terms of the Fermi points. To this end, we turn to the total particle density n and magnetization \bar{m} per site based on eq. (4.0.5)

$$n = -\frac{\partial f}{\partial \mu} = -\int_{-Q_u}^{Q_u} \frac{dk}{2\pi} \frac{\partial \epsilon^u}{\partial \mu} - \beta_1 \int_{-Q_b}^{Q_b} \frac{d\Lambda}{2\pi} \frac{\partial \epsilon^b}{\partial \mu}, \quad (4.1.15)$$

$$\bar{m} = -\frac{\partial f}{\partial B} = -\int_{-Q_u}^{Q_u} \frac{dk}{2\pi} \frac{\partial \epsilon^u}{\partial B} - \beta_1 \int_{-Q_b}^{Q_b} \frac{d\Lambda}{2\pi} \frac{\partial \epsilon^b}{\partial B}. \quad (4.1.16)$$

Here we redefine $\bar{m} = 2m$ according to the original TBA equations. In order to get closed forms for these two properties, we first take partial derivatives of eqs. (4.1.1) and (4.1.2) with respect to $\tilde{\mu}$ and \tilde{B} , respectively. Similarly, $\tilde{\mu} = \mu/u^2$ and $\tilde{B} = B/u^2$. Both sets of equations for these derivatives look similar to the zero temperature TBA equations except the different driving terms. It is easy to rewrite these integral equations in terms of the vectorial forms similar to eqs. (4.1.3) and (4.1.4)

$$\frac{\partial \tilde{\epsilon}^u}{\partial \tilde{\mu}} = \frac{\partial \tilde{V}^u}{\partial \tilde{\mu}} - \mathbf{A}^1(\tilde{Q}_b) \frac{\partial \tilde{\epsilon}^b}{\partial \tilde{\mu}}, \quad (4.1.17)$$

$$\frac{\partial \tilde{\epsilon}^b}{\partial \tilde{\mu}} = \frac{\partial \tilde{V}^b}{\partial \tilde{\mu}} - \mathbf{A}^1(\tilde{Q}_u) \frac{\partial \tilde{\epsilon}^u}{\partial \tilde{\mu}} - \mathbf{A}^2(\tilde{Q}_b) \frac{\partial \tilde{\epsilon}^b}{\partial \tilde{\mu}}, \quad (4.1.18)$$

$$\frac{\partial \tilde{\epsilon}^u}{\partial \tilde{B}} = \frac{\partial \tilde{V}^u}{\partial \tilde{B}} - \mathbf{A}^1(\tilde{Q}_b) \frac{\partial \tilde{\epsilon}^b}{\partial \tilde{B}}, \quad (4.1.19)$$

$$\frac{\partial \tilde{\epsilon}^b}{\partial \tilde{B}} = \frac{\partial \tilde{V}^b}{\partial \tilde{B}} - \mathbf{A}^1(\tilde{Q}_u) \frac{\partial \tilde{\epsilon}^u}{\partial \tilde{B}} - \mathbf{A}^2(\tilde{Q}_b) \frac{\partial \tilde{\epsilon}^b}{\partial \tilde{B}}, \quad (4.1.20)$$

where the driving terms are $\frac{\partial \bar{V}^u}{\partial \mu} = [-1, 0, \dots]^{Tr}$, $\frac{\partial \bar{V}^b}{\partial \mu} = [-2, 0, \dots]^{Tr}$, $\frac{\partial \bar{V}^u}{\partial B} = [-1, 0, \dots]^{Tr}$, and $\frac{\partial \bar{V}^b}{\partial B} = [0, 0, \dots]^{Tr}$.

The resolving procedure for the above vectorial equations of partial derivatives are rather similar to that for the TBA equations. These solutions together with eq. (4.1.15) result in

$$\tilde{n} = \frac{1}{\pi} [\tilde{Q}_u, \beta_1 \tilde{Q}_b] (\mathbf{I} - \mathbf{T} + \mathbf{T}^2) \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad (4.1.21)$$

$$\tilde{m} = \frac{1}{\pi} [\tilde{Q}_u, \beta_1 \tilde{Q}_b] (\mathbf{I} - \mathbf{T} + \mathbf{T}^2) \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad (4.1.22)$$

where $\tilde{n} = n/|u|$ and $\tilde{m} = \bar{m}/|u|$.

Finally, if we define $n_u = \bar{m}$ and $n_b = (n - n_u)/2$ as the densities for the excess fermions and bound pairs respectively, eqs. (4.1.21) and (4.1.22) can be summarized as

$$\begin{bmatrix} \tilde{n}_u \\ \tilde{n}_b \end{bmatrix} = \frac{1}{\pi} (\mathbf{I} - \mathbf{T} + \mathbf{T}^2)^{Tr} \begin{bmatrix} \tilde{Q}_u \\ \beta_1 \tilde{Q}_b \end{bmatrix}. \quad (4.1.23)$$

Here similarly $\tilde{n}^\gamma = n^\gamma/|u|$ ($\gamma = u, b$).

A reverse of eq. (4.1.23) gives the cut-off momenta

$$\begin{aligned} Q_u &\approx \pi n_u \sum_{j=0}^5 \left(\frac{2n_b}{|u|} \right)^j, \\ Q_b &\approx \frac{\pi n_b}{\beta_1} \sum_{n=j}^5 \left[\frac{2n_u + n_b}{\beta_1 |u|} \right]^j. \end{aligned} \quad (4.1.24)$$

For the next step, substituting eq. (4.1.24) into eq. (4.1.14) leads to separating the ground state energy per site into the energies of excess fermions and bound pairs, with result

$$e = e_u + e_b + e_{bind}. \quad (4.1.25)$$

Here e_{bind} is the binding energy and the subscripts u and b denote the excess fermions and bound pairs, respectively. The terms are given explicitly by

$$e_u = \frac{\pi^2}{3} n_u^3 \left[1 + 2 \left(\frac{2n_b}{|u|} \right) + 3 \left(\frac{2n_b}{|u|} \right)^2 \right], \quad (4.1.26)$$

$$e_b = \frac{\pi^2}{3} \frac{\alpha_1 n_b^3}{\beta_1^2} \left[1 + 2 \left(\frac{2n_u + n_b}{\beta_1 |u|} \right) + 3 \left(\frac{2n_u + n_b}{\beta_1 |u|} \right)^2 \right], \quad (4.1.27)$$

$$e_{bind} = -(2u+2)n_u - 4\left(u + \sqrt{u^2+1}\right)n_b. \quad (4.1.28)$$

We define a dimensionless interaction strength $\gamma_s = 2|u|/n_s$ ($s = u, b$), and then using the relation

$$K_s = \pi / \sqrt{3e(\gamma_s) - 2\gamma_s \frac{de(\gamma_s)}{d\gamma_s} + \frac{1}{2}\gamma_s^2 \frac{d^2e(\gamma_s)}{d\gamma_s^2}}, \quad (4.1.29)$$

the Luttinger parameters for the excess fermions and bound pairs can be directly worked out to be

$$\begin{aligned} K_1 &= 1, \\ K_2 &= 2\sqrt{2} \frac{\beta_1}{\sqrt{\alpha_1}} \left[1 - \frac{2}{\beta_1 \gamma_2} + \frac{1}{(\beta_1 \gamma_2)^2} \right]. \end{aligned} \quad (4.1.30)$$

We note that the Luttinger parameter K_2 in the fully paired phase V depends explicitly on the lattice parameters α_1 and β_1 . This behavior is different from the constant value $K_2 = 4$ for the bound pairs phase of the strongly attractive 1D $SU(2)$ Fermi gas [91]. In the limits $u \rightarrow 0$ and $n_s/|u|$ ($s = u, b$) small, the lattice parameters $\alpha_1 \rightarrow 2$, $\beta_1 \rightarrow 2$, see fig. 4.1. Thus we have $K_2 = 4$ which is the same as for the $SU(2)$ Fermi gas. The two limits $u \rightarrow 0$ and $n_s/|u| \ll 1$ represent the lattice-gas mapping between 1D attractive Hubbard model and $SU(2)$ Fermi gas [18].

4.2 Additivity Rule

By virtue of eqs. (4.1.12) and (4.1.23), we find that for the low density case, the chemical potentials for the unpaired fermions and pairs are given explicitly by

$$\mu^u = \pi n_u^2 A_1^2 + \frac{4\pi^2 \alpha_1}{3\beta_1^3 |u|} n_b^3 A_2^2, \quad (4.2.1)$$

$$\mu^b = \pi^2 \frac{n_b^2}{\beta_1^2} A_2^2 + \frac{4\pi^2}{3\alpha_1 |u|} n_u^3 A_1^3 + \frac{2\pi^2}{3\beta_1^3 |u|} n_b^3 A_2^3, \quad (4.2.2)$$

with

$$\begin{aligned} A_1 &= 1 + \frac{2n_b}{|u|} + \left(\frac{2n_b}{|u|} \right)^2, \\ A_2 &= 1 + \frac{2n_u + n_b}{\beta_1 |u|} + \left(\frac{2n_u + n_b}{\beta_1 |u|} \right)^2, \end{aligned} \quad (4.2.3)$$

which indicate interacting effects among pairs and unpaired fermions. Specifically, A_1 and A_2 represent the statistical parameters for ideal anyonic particles of single atoms and pairs, respectively. The effective chemical potentials eqs. (4.2.1) and (4.2.2) reveal that the thermodynamic quantities could be separable, i.e. the total is equal to a sum of the effective thermodynamic quantities of two individual constituents. In the next stage, we further derive the additivity rules for the compressibility and susceptibility.

For the compressibility, using the standard thermodynamic relation $\kappa = \left(\frac{\partial n}{\partial \mu}\right)_B$, the derivatives of the density and effective chemical potentials for fixed magnetic field can be respectively expressed as

$$\begin{aligned} dn &= dn_u + 2 dn_b, \\ d\mu^u &= \frac{\alpha_1}{2} d\mu^b = d\mu. \end{aligned} \quad (4.2.4)$$

Inserting these relations into the definition of compressibility $\kappa = \left(\frac{\partial n}{\partial \mu}\right)_B = \frac{dn_u + 2dn_b}{d\mu}$, it follows that

$$\kappa_c = \kappa_u + \frac{2}{\alpha_1} \kappa_b. \quad (4.2.5)$$

Here the effective compressibilities of excess fermions and bound pairs are respectively defined by

$$\kappa_u = \left(\frac{\partial n_u}{\partial \mu^u}\right)_B, \quad (4.2.6)$$

$$\kappa_b = 2 \left(\frac{\partial n_b}{\partial \mu^b}\right)_B. \quad (4.2.7)$$

In light of eqs. (4.0.3) and (4.0.4), they can be further expressed as

$$\kappa_u = \left(\frac{\partial n_u}{\partial \mu^u}\right)_B = \left(\frac{\partial n_u}{\partial \mu}\right)_B, \quad (4.2.8)$$

$$\kappa_b = 2 \left(\frac{\partial n_b}{\partial \mu^b}\right)_B = \alpha_1 \left(\frac{\partial n_b}{\partial \mu}\right)_B. \quad (4.2.9)$$

We now proceed to derive the explicit expressions for the effective compressibility in terms of densities of bound pairs and excess fermions. Apparently, the densities of bound pairs and excess fermions rely on the chemical potential and the magnetic field, and vice versa. Hence the total derivatives of n_u and n_b with

respect to μ and B tell us

$$\begin{bmatrix} dn_u \\ dn_b \end{bmatrix} = \begin{bmatrix} \left(\frac{\partial n_u}{\partial \mu}\right)_B & \left(\frac{\partial n_u}{\partial B}\right)_\mu \\ \left(\frac{\partial n_b}{\partial \mu}\right)_B & \left(\frac{\partial n_b}{\partial B}\right)_\mu \end{bmatrix} \begin{bmatrix} d\mu \\ dB \end{bmatrix}, \quad (4.2.10)$$

and the total derivatives of μ and B with respect to n_u and n_b read

$$\begin{bmatrix} d\mu \\ dB \end{bmatrix} = \begin{bmatrix} \left(\frac{\partial \mu}{\partial n_u}\right)_{n_b} & \left(\frac{\partial \mu}{\partial n_b}\right)_{n_u} \\ \left(\frac{\partial B}{\partial n_u}\right)_{n_b} & \left(\frac{\partial B}{\partial n_b}\right)_{n_u} \end{bmatrix} \begin{bmatrix} dn_u \\ dn_b \end{bmatrix}. \quad (4.2.11)$$

We resolve the set of linear equations in eq. (4.2.11) and obtain

$$\begin{bmatrix} dn_u \\ dn_b \end{bmatrix} = \frac{1}{J} \begin{bmatrix} \left(\frac{\partial B}{\partial n_b}\right)_{n_u} & -\left(\frac{\partial \mu}{\partial n_b}\right)_{n_u} \\ -\left(\frac{\partial B}{\partial n_u}\right)_{n_b} & \left(\frac{\partial \mu}{\partial n_u}\right)_{n_b} \end{bmatrix} \begin{bmatrix} d\mu \\ dB \end{bmatrix}, \quad (4.2.12)$$

comparing which with eq. (4.2.10) yields

$$\begin{aligned} \left(\frac{\partial n_u}{\partial \mu}\right)_B &= \frac{1}{J} \left(\frac{\partial B}{\partial n_b}\right)_{n_u}, \\ \left(\frac{\partial n_b}{\partial \mu}\right)_B &= -\frac{1}{J} \left(\frac{\partial B}{\partial n_u}\right)_{n_b}. \end{aligned} \quad (4.2.13)$$

Here J is the Jacobian determinant expressed by

$$\begin{aligned} J &= \left(\frac{\partial \mu}{\partial n_u}\right)_{n_b} \left(\frac{\partial B}{\partial n_b}\right)_{n_u} - \left(\frac{\partial B}{\partial n_u}\right)_{n_b} \left(\frac{\partial \mu}{\partial n_b}\right)_{n_u} \\ &= -\frac{\alpha_1}{2} \left[\left(\frac{\partial \mu^u}{\partial n_u}\right)_{n_b} \left(\frac{\partial \mu^b}{\partial n_b}\right)_{n_u} - \left(\frac{\partial \mu^b}{\partial n_u}\right)_{n_b} \left(\frac{\partial \mu^u}{\partial n_b}\right)_{n_u} \right]. \end{aligned} \quad (4.2.14)$$

Similarly, the magnetic field is dependent on the effective chemical potentials while the latter is dependent on densities of bound pairs and excess fermions. Therefore by application of the chain rule we have

$$\begin{aligned} \left(\frac{\partial B}{\partial n_u}\right)_{n_b} &= \left(\frac{\partial \mu^u}{\partial n_u}\right)_{n_b} - \frac{\alpha_1}{2} \left(\frac{\partial \mu^b}{\partial n_u}\right)_{n_b}, \\ \left(\frac{\partial B}{\partial n_b}\right)_{n_u} &= \left(\frac{\partial \mu^u}{\partial n_b}\right)_{n_u} - \frac{\alpha_1}{2} \left(\frac{\partial \mu^b}{\partial n_b}\right)_{n_u}. \end{aligned} \quad (4.2.15)$$

Since we have obtained the explicit expression of μ_r in terms of n_s ($r, s = u, b$)

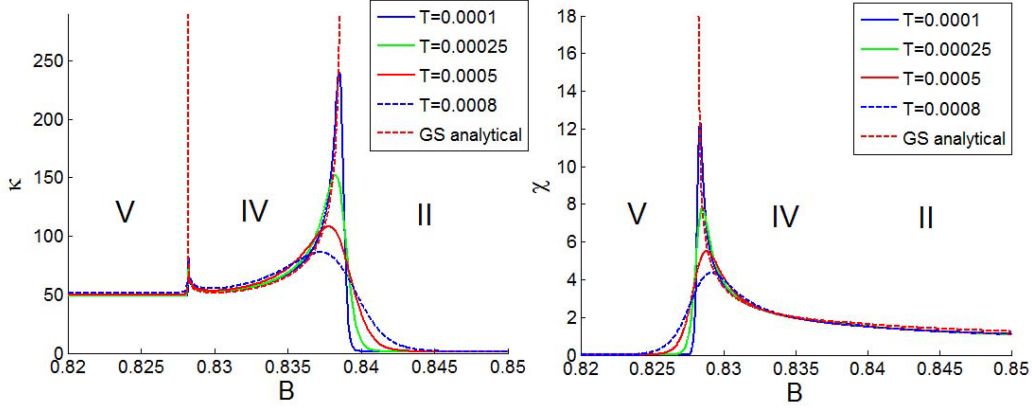


Figure 4.2: A plot of compressibility κ and spin susceptibility χ vs magnetic field B for the 1D attractive Hubbard model with $u = -1$ and $\mu = -0.8282$. The red dashed lines show the result obtained from the additivity rules eqs. (4.2.5) and (4.2.20). At low temperatures, all compressibility and susceptibility curves collapse into the zero temperature ones obeying the additivity rules. In the vicinity of the critical points such free-Fermi liquid nature breaks down.

in eqs. (4.2.1) and (4.2.2), our goal of the effective compressibilities is easy to achieve, with the results

$$\kappa_u = \frac{\pi^2}{J} \left[-\frac{\alpha_1 n_b}{\beta_1^2} - \frac{4\alpha_1 n_u n_b}{|u|\beta_1^3} + \frac{4n_u^2}{|u|} - \frac{4n_u^3}{u^2} + \frac{24n_u^2 n_b}{u^2} - \frac{12\alpha_1 n_u^2 n_b}{u^2 \beta_1^4} + \frac{6\alpha_1 n_b^3}{u^2 \beta_1^4} \right], \quad (4.2.16)$$

$$\kappa_b = -\frac{2\alpha_1 \pi^2}{J} \left[n_u - \frac{n_u^2}{|u|} + \frac{4n_u n_b}{|u|} - \frac{6n_u^2 n_b}{u^2} - \frac{\alpha_1 n_b^2}{u\beta_1^3} + \frac{12n_u n_b^2}{u^2} - \frac{6\alpha_1 n_u n_b^2}{u^2 \beta_1^4} \right], \quad (4.2.17)$$

$$J = -\frac{2\pi^4 \alpha_1}{\beta_1^2} n_u n_b \left[1 + \frac{4n_u}{|u|\beta_1} + \frac{12n_u^2}{u^2 \beta_1^2} + \frac{4n_b}{|u|} + \frac{4n_b}{|u|\beta_1} + \frac{24n_u n_b}{u^2 \beta_1^2} + \frac{8n_u n_b}{u^2 \beta_1} + \frac{12n_b^2}{u^2} + \frac{10n_b^2}{u^2 \beta_1^2} + \frac{16n_b^2}{u^2 \beta_1} \right]. \quad (4.2.18)$$

For the susceptibility defined in the canonical ensemble, $\tilde{\chi} = \left(\frac{\partial \bar{m}}{\partial B} \right)_n$, it is straightforward to see

$$\begin{aligned} dn &= dn_u + 2dn_b = 0, \\ dB &= d\mu^u - \frac{\alpha_1}{2} d\mu^b, \end{aligned} \quad (4.2.19)$$

where the second equation for dB arises from eqs. (4.0.3) and (4.0.4).

The additivity rule for the susceptibility states that

$$\frac{1}{\bar{\chi}} = \frac{1}{\bar{\chi}_u} + \frac{\alpha_1}{2} \frac{1}{\bar{\chi}_b}, \quad (4.2.20)$$

where the effective susceptibilities for excess fermions and bound pairs are respectively defined by

$$\bar{\chi}_u = \left(\frac{\partial n_u}{\partial \mu^u} \right)_n, \quad (4.2.21)$$

$$\bar{\chi}_b = 2 \left(\frac{\partial n_b}{\partial \mu^b} \right)_n. \quad (4.2.22)$$

It is easy to carry out the calculation for this relation,

$$\frac{1}{\bar{\chi}} = \left(\frac{\partial \bar{m}}{\partial B} \right)_n^{-1} = \frac{dB}{dn_u} = \frac{d\mu^u}{dn_u} - \frac{\alpha_1}{2} \frac{d\mu^b}{dn_u} = \frac{d\mu^u}{dn_u} + \frac{\alpha_1}{2} \frac{d\mu^b}{2dn_b} = \frac{1}{\bar{\chi}_u} + \frac{\alpha_1}{2} \frac{1}{\bar{\chi}_b}. \quad (4.2.23)$$

The derivation for explicit expressions of the effective susceptibilities is rather simple. Due to the fixed total particle density, one confirms that $dn_u + 2dn_b = 0$, and thus the total derivative of the effective chemical potentials with respect to n_γ ($\gamma = u, b$) is

$$\begin{aligned} d\mu^u &= \left[\left(\frac{\partial \mu^u}{\partial n_u} \right)_{n_b} - \frac{1}{2} \left(\frac{\partial \mu^u}{\partial n_b} \right)_{n_u} \right] dn_u, \\ d\mu^b &= -2 \left[\left(\frac{\partial \mu^b}{\partial n_u} \right)_{n_b} - \frac{1}{2} \left(\frac{\partial \mu^b}{\partial n_b} \right)_{n_u} \right] dn_b, \end{aligned} \quad (4.2.24)$$

which indicates that

$$\begin{aligned} \bar{\chi}_u &= \frac{1}{\left(\frac{\partial \mu^u}{\partial n_u} \right)_{n_b} - \frac{1}{2} \left(\frac{\partial \mu^u}{\partial n_b} \right)_{n_u}}, \\ \bar{\chi}_b &= \frac{-1}{\left(\frac{\partial \mu^b}{\partial n_u} \right)_{n_b} - \frac{1}{2} \left(\frac{\partial \mu^b}{\partial n_b} \right)_{n_u}}. \end{aligned} \quad (4.2.25)$$

Taking account of eqs. (4.2.1) and (4.2.2), one then derives the results

$$\bar{\chi}_u = \frac{1/(2\pi^2)}{n_u - \frac{n_u^2}{|u|} + \frac{4n_u n_b}{|u|} - \frac{6n_u^2 n_b}{u^2} - \frac{\alpha_1 n_b^2}{|u| \beta_1^3} + \frac{12n_u n_b^2}{u^2} - \frac{6\alpha_1 n_u n_b^2}{u^2 \beta_1^4}},$$

$$\bar{\chi}_b = \frac{1/\pi^2}{\frac{n_b}{\beta_1^2} - \frac{4n_u^2}{|u|\alpha_1} + \frac{4n_u^3}{u^2\alpha_1} + \frac{4n_un_b}{|u|\beta_1^3} - \frac{24n_u^2n_b}{u^2\alpha_1} + \frac{12n_u^2n_b}{u^2\beta_1^4} - \frac{6n_b^3}{u^2\beta_1^4}}. \quad (4.2.26)$$

Similar to the observation concerning TLL parameters, the additivity rules for the 1D attractive Hubbard model also reduce to those for the $SU(2)$ Fermi gas through the lattice-gas mapping [91],

$$\kappa = \kappa_u + \kappa_b, \quad \frac{1}{\chi} = \frac{1}{\chi_u} + \frac{1}{\chi_b}. \quad (4.2.27)$$

The simple additivity nature of the thermodynamics at low temperatures characterizes the universal low energy physics of the FFLO-like state of the 1D attractive Hubbard model. In this sense the additivity rules reflect a universal nature of the multicomponent TLL in 1D. It is reminiscent of two independent Fermi liquids. The simple additivity rule thus reveals the significant free-Fermi liquid nature of the FFLO phase. Meanwhile they also reveal a free-Fermi liquid nature in the phase of the multiple states.

The macroscopic magnetic properties in the FFLO-like phase show the properties of the ordinary higher-dimensional Fermi liquid, see fig. 4.3. This figure shows that in the Fermi liquid region the magnetization is nearly temperature independent. In the non-Fermi liquid region thermal fluctuations gradually overwhelm quantum fluctuations. Thus the magnetization has a uniform temperature dependence for different magnetic fields, indicating paramagnetism. Such Fermi liquid features have been found in the spin compound CupzN [86] and the heavy fermion material YbNi_4P_2 [87]. Recently the study of Fermi and non-Fermi liquids in 1D has received significant interest [88, 98, 99].

4.3 Wilson Ratio

The Wilson ratio is a dimensionless ratio defined by the compressibility κ_c or susceptibility χ over the specific heat divided by the temperature T

$$R_W^\kappa = \frac{\pi^2 k_B^2}{3} \frac{\kappa_c}{C_v/T}, \quad (4.3.1)$$

$$R_W^\chi = \frac{4}{3} \left(\frac{\pi k_B}{\mu_B g_L} \right)^2 \frac{\chi}{C_v/T}, \quad (4.3.2)$$

where k_B is the Boltzmann constant, μ_B the Bohr magneton and g_L the Lande factor. The Wilson ratio characterises the competition between quantum fluctuations and thermal fluctuations.

We find that the ratio R_W^κ is capable of distinguishing all phases of quantum

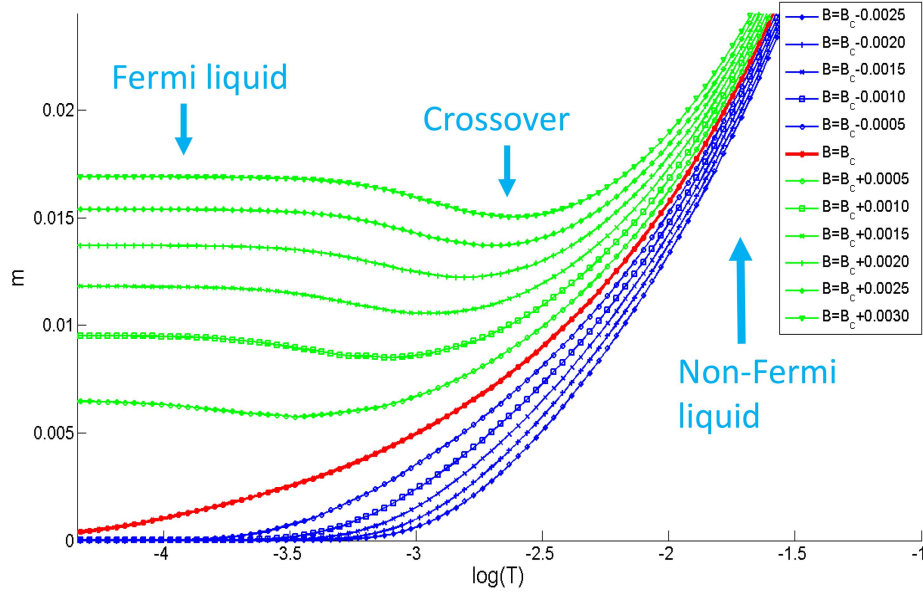


Figure 4.3: Numerical results for the magnetization vs logarithm of the temperature for different magnetic fields. Here we have set a fixed chemical potential $\mu = -0.14$ and interaction strength $u = -7$. For magnetic field $B > B_c = 12.11065$ (phase IV), three regions are clearly displayed: Fermi liquid region at low temperatures, non-Fermi liquid region at higher temperatures, and a crossover in between. For magnetic field $B < B_c$ (phase V), the magnetization displays the gapped nature of a non-Fermi liquid phase.

states, including the FFLO-like state in the phase diagram fig. 4.4. Indeed, the phase boundaries determined by the Wilson ratio eq. (4.3.1) coincide with the ones determined by the zero temperature TBA equations, see fig. 2.2. However, the usual Wilson ratio R_W^χ does not distinguish between the vacuum and the pure paired phase V for $T \rightarrow 0$ due to the vanishing susceptibility in both phases.

We observe that an enhancement of this ratio occurs near a phase transition. It gives a finite value at the critical point, unlike the divergent values of compressibility and susceptibility for $T \rightarrow 0$, see fig. 4.5. Hence the Wilson ratio offers an efficient method to determine the low temperature phase diagram.

The phases IV and V in fig. 4.4 reveal significant features, namely Fermi liquids and free-fermion quantum criticality. A constant Wilson ratio implies that the two types of fluctuations are on an equal footing, regardless of the microscopic details of the underlying many-body systems. Using the explicit expressions for the compressibility eq. (4.2.5) and susceptibility eq. (4.2.20), we may calculate the Wilson ratio. The compressibility Wilson ratio R_W^κ is determined by

$$R_W^\kappa = \frac{\pi^2 k_B^2}{3} \frac{\kappa}{C_v/T}$$

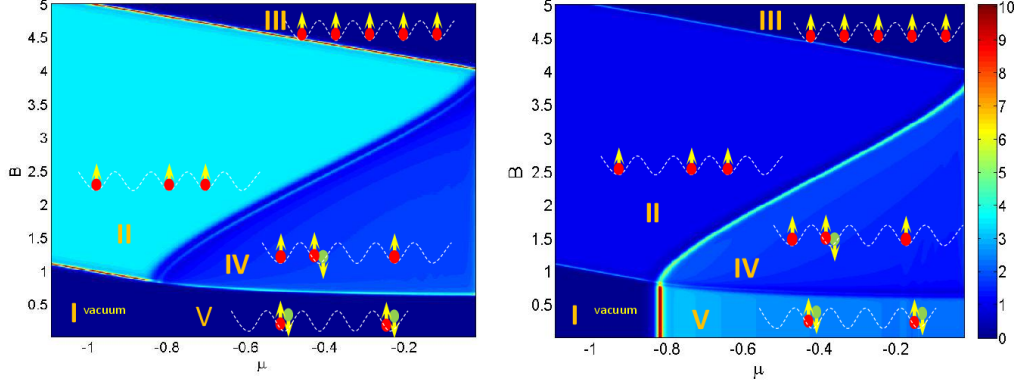


Figure 4.4: Finite temperature phase diagrams obtained from contour plots of the Wilson ratios. The left and right plots are determined from the susceptibility Wilson ratio R_W^χ and from the compressibility Wilson ratio R_W^κ , respectively. Here $u = -1$ and $T = 0.001$. The red balls and green balls represent up spin and down spin respectively. Both diagrams agree well with the zero temperature phase diagram fig. 2.2, despite the fact that the left plot cannot distinguish phase I and V.

$$= \pi \left(\kappa_u + \frac{2}{\alpha_1} \kappa_b \right) / \left(\frac{1}{v_1} + \frac{1}{v_2} \right), \quad (4.3.3)$$

where we have used eq. (3.4.1) to calculate the specific heat and set the Boltzmann constant $k_B = 1$. This Wilson ratio vanishes in both phases I (vacuum) and III (half-filling phase). In the limit $n/|u| \rightarrow 0$ the compressibility Wilson ratio for phases II and V are respectively given by

$$R_W^\kappa = 1, \quad (4.3.4)$$

$$R_W^\kappa = 2\sqrt{2}\beta_1/\sqrt{\alpha_1}, \quad (4.3.5)$$

which turn out to be the same as that for the 1D strongly attractive $SU(2)$ Fermi gas [91] when the limit $u \rightarrow 0$ is applied, i.e., $R_W^\kappa = 1$ and $R_W^\kappa = 4$.

Last but not least, we explicitly express the sound velocities in the low-density regime as follows,

$$\begin{aligned} v^u &\approx 2\pi n_u \left[1 + 4 \frac{n_b}{|u|} + 12 \left(\frac{n_b}{|u|} \right)^2 \right], \\ v^b &\approx \pi n_b \frac{\sqrt{2\alpha_1}}{\beta_1} \left[1 + \frac{1}{\beta_1} \left(2 \frac{n_u}{|u|} + \frac{n_b}{|u|} \right) + \frac{3}{\beta_1^2} \left(2 \frac{n_u}{|u|} + \frac{n_b}{|u|} \right)^2 \right], \end{aligned} \quad (4.3.6)$$

which are derived by using the formula $v^\gamma = \sqrt{\frac{L}{m_p n_\gamma r_\gamma} \frac{\partial^2 E_\gamma}{\partial L^2}}$ with $r_u = 1$ and $r_b = 2$, where the particle mass has been rescaled as $m_p = \frac{1}{2}$ and $E_\gamma = e_\gamma L$. See e_γ in

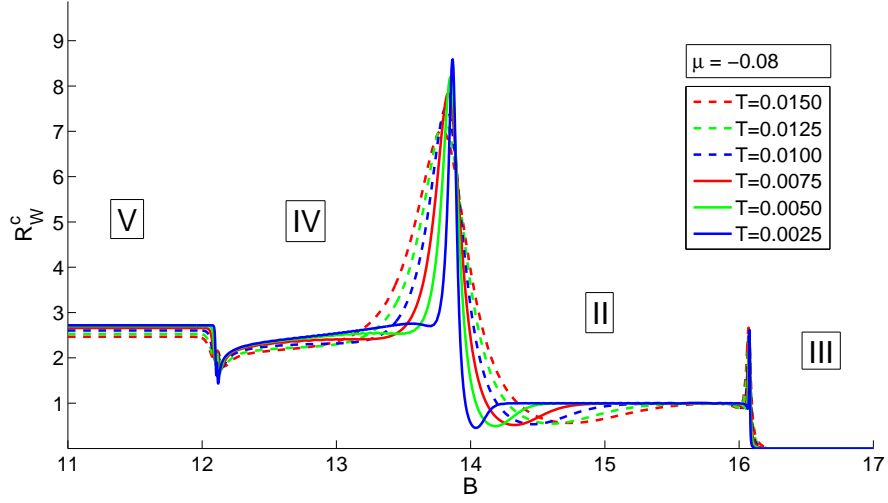


Figure 4.5: Wilson Ratio R_W^c vs magnetic field for fixed $\mu = -0.08$ and $u = -7$. The sharp peaks at phase transitions distinguish different quantum phases V, IV, II and III, respectively. The constant values of the ratio show Fermi liquid nature in these phases.

eqs. (4.1.26) and (4.1.27).

4.4 Conclusion

In conclusion to this chapter, we have studied the Luttinger parameters, additivity rule and Wilson ratio in the FFLO-like phase. Through the introduction of two effective chemical potentials for the bound pairs and the excess fermions, we successfully separate the ground state energy into three parts, using which the Luttinger parameters of each constituent are obtained. Moreover, these two effective chemical potentials help us prove the additivity rule for the thermodynamic properties, compressibility and susceptibility. The additivity rule describes the macroscopic behavior of the FFLO-like phase, reminiscent of two independently Fermi liquids. The effective compressibilities and susceptibilities of these two liquids (bound pairs and excess fermions) are explicitly given in terms of the densities n_u and n_b . These results in turn afford us the expression of compressibility Wilson ratio, which has an enhancement in the vicinity of quantum critical region while changing slowly away from the critical points. Thereby it clearly distinguishes the five phases for low temperature.

We observe that the lattice effect of the Hubbard model is prominent in terms of two parameters α_1 and β_1 . In the limit $u \rightarrow 0$ and $n/|u| \ll 1$, the Luttinger parameters, additivity rule and Wilson ratio of the 1D attractive Hubbard model

can be mapped into the case of the 1D strongly attractive $SU(2)$ Fermi gas.

FFLO Signature for the 1D Attractive Hubbard Model

The Hubbard model with attractive interaction is considered as a promising candidate to explain high T_c superconductivity. To this end, making clear the pairing mechanism in 1D is of significant importance. According to the Bardeen-Cooper-Schrieffer (BCS) theory, a Cooper pair is formed by electrons with opposite spins and momenta and the total momentum zero. This balance between Fermi energies breaks down in the presence of strong magnetic field, so that a novel superconductive state - the FFLO state - appears [103, 104]. The FFLO pair carries non-zero center-of-mass momentum, originating from the strong magnetic field. The superconducting order parameter and density of spins in the FFLO state exhibit a periodic oscillation in the spatial coordinate. The experimental observation of the FFLO state in various materials has been sought for decades. Within this scenario, more evidence has been found in heavy-fermion systems [105, 106].

The pair mechanism in the 1D attractive Hubbard model has been investigated [56–58]. Cooper pairs in 1D exist in the region $B < B_{c1}$ below the critical magnetic field B_{c1} , where the average distance between pairs is much larger than the average pair size. This means that the single particle Green's function decays exponentially and singlet pair correlation function decays as a power of distance. Once exceeding the critical magnetic field ($B > B_{c1}$), the field begins to break up the pairs, and both the above correlation functions decay as a power of distance.

The FFLO state in 1D attractive Hubbard model has been extensively studied by various methods, such as the numerical approaches of the density matrix renormalized group (DMRG) [59–63], and quantum Monte Carlo (QMC) [64, 65]. It has been found that the pair correlation complies with a power-law decay, i.e., $n_{pair} \propto \cos(k_{FFLO}|x|)/|x|^\alpha$, and its corresponding momentum distribution has peaks in the position of $k_{FFLO} = \pi(n_\uparrow - n_\downarrow)$ [59]. In light of the available numerical studies, a theoretical confirmation of the FFLO state in the 1D attractive Hubbard model is still desirable.

In this chapter we focus on asymptotic correlation functions of the 1D at-

tractive Hubbard model in the partially polarized phase IV, which is dominated by a universality class of two-component Tomonaga-Luttinger liquid (TLL) in low-temperature regimes, composed of bound pairs and of unpaired fermions, respectively.

5.1 Ground State and Finite Size Correction

Prior to later discussion, we add the magnetic field and chemical potential to the hamiltonian as follows

$$\begin{aligned}
 H = & - \sum_{j=1}^L \sum_{a=\uparrow,\downarrow} \left(c_{j,a}^\dagger c_{j+1,a} + c_{j+1,a}^\dagger c_{j,a} \right) + u \sum_{j=1}^L (1 - 2n_{j,\uparrow}) (1 - 2n_{j,\downarrow}) \\
 & - B \sum_{j=1}^L (n_{\uparrow,j} - n_{\downarrow,j}) - \mu \sum_{j=1}^L (n_{\uparrow,j} + n_{\downarrow,j}).
 \end{aligned} \tag{5.1.1}$$

The ground state of the 1D attractive Hubbard model is filled with single real k and k - Λ string of length one (i.e., $m = 1$). The Λ - Λ strings are suppressed due to the ferromagnetic ordering. This means that the quasimomenta of fermions in bound pairs and the excess unpaired fermions can be respectively written as $k_{\eta,\pm}^b = \arcsin(k_\eta^b \pm i|u|)$ and k_β^u , with $\eta = 1, 2, \dots, M$. For simplicity, we have denoted k as k_β^u , and Λ as k_η^b . Substituting above simplified string hypothesis into eqs. (2.2.48) and (2.2.49) and then taking logarithms, we arrive at the discrete Bethe ansatz equations for the ground state

$$\frac{2\pi}{L} I_j^u = k_j^u - \frac{1}{L} \sum_{\alpha=1}^{N^b} \theta \left(\frac{\sin k_j^u - k_\alpha^b}{|u|} \right), \tag{5.1.2}$$

$$\frac{2\pi}{L} I_\gamma^b = 2 \operatorname{Re} \left[\arcsin(k_\gamma^b + i|u|) \right] - \frac{1}{L} \sum_{j=1}^{N^u} \theta \left(\frac{k_\gamma^b - \sin k_j^u}{|u|} \right) + \frac{1}{L} \sum_{\beta=1}^{N^b} \theta \left(\frac{k_\gamma^b - k_\beta^b}{2|u|} \right), \tag{5.1.3}$$

where N^u (N^b) is the number of unpaired fermions (bound pairs) satisfying $N^u + 2N^b = N$, and the quantum numbers take the values

$$I_j^u \in \mathbb{Z} + \frac{N^b}{2}, \quad I_j^b \in \mathbb{Z} + \frac{N^u + N^b + 1}{2}. \tag{5.1.4}$$

The ground state energy and momentum are explicitly expressed as

$$E = \sum_{j=1}^{N^u} (-2 \cos k_j^u - \mu - 2u - B) + \sum_{\gamma=1}^{N^b} \left[-2\mu - 4u - 4 \operatorname{Re} \sqrt{1 - (k_\gamma^b + i|u|)^2} \right], \quad (5.1.5)$$

$$P = \sum_{j=1}^{N^u} k_j^u + \sum_{\eta}^{N^b} (k_{\eta,+}^b + k_{\eta,-}^b). \quad (5.1.6)$$

We have defined the counting functions in eq. (2.3.11), and now for convenience we denote $y_L^u(k_j^u) = \frac{2\pi l_j^u}{L}$, and $y_L^b(k_\beta^b) = \frac{2\pi l_\beta^b}{L}$, which satisfy

$$\begin{aligned} \rho_L^u(k^u) &= \frac{1}{2\pi} \frac{dy_L^u(k^u)}{dk^u} \\ &= \frac{1}{2\pi} - \frac{1}{L} \sum_{j=1}^{N^b} a_1(\sin k^u - k_j^b) \cos k^u, \end{aligned} \quad (5.1.7)$$

$$\begin{aligned} \rho_L^b(k^b) &= \frac{1}{2\pi} \frac{dy_L^b(k^b)}{dk^b} \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} dk a_1(k^b - \sin k) - \frac{1}{L} \sum_{j=1}^{N^u} a_1(k^b - \sin k_j^u) - \frac{1}{L} \sum_{j=1}^{N^b} a_2(k^b - k_j^b), \end{aligned} \quad (5.1.8)$$

where $\rho_L^\gamma(k^\gamma)$ ($\gamma = u, b$) is the root density of corresponding quasimomentum.

To obtain the finite-size correction to the above root densities when $L \gg 1$, we utilize the Euler-MacLaurin formula and obtain

$$\begin{aligned} \rho_L^u(k^u) &= \frac{1}{2\pi} - \cos k^u \int_{Q_-^b}^{Q_+^b} dk^b a_1(\sin k^u - k^b) \rho_L^b(k^b) \\ &\quad - \frac{1}{24L^2} \cos k^u \left. \frac{a_1'(\sin k^u - k^b)}{\rho_L^b(k^b)} \right|_{k^b=Q_-^b}^{k^b=Q_+^b} + \text{h.o.c.}, \end{aligned} \quad (5.1.9)$$

$$\begin{aligned} \rho_L^b(k^b) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} dk a_1(k^b - \sin k) - \int_{Q_-^u}^{Q_+^u} dk^u a_1(k^b - \sin k^u) \rho_L^u(k^u) - \int_{Q_-^b}^{Q_+^b} dk a_2(k^b - k) \rho_L^b(k) \\ &\quad - \frac{1}{24L^2} \left[\left. \frac{\cos k^u a_1'(k^b - \sin(k^u))}{\rho_L^u(k^u)} \right|_{k^u=Q_-^u}^{k^u=Q_+^u} + \left. \frac{a_2'(k^b - k)}{\rho_L^b(k)} \right|_{k=Q_-^b}^{k=Q_+^b} \right] + \text{h.o.c.}, \end{aligned} \quad (5.1.10)$$

where $a_n'(x)$ is the derivative of $a_n(x)$, and Q_\pm^γ ($\gamma = u, b$) denotes the Fermi point.

In order to derive the finite-size corrections to the ground state and low-lying

excitations, we rewrite the TBA equations in the zero temperature limit as

$$\varepsilon^u(k^u) = -2\cos k^u - \mu - 2u - B - \int_{Q_-^b}^{Q_+^b} dk^b a_1(\sin k^u - k^b) \varepsilon^b(k^b) \quad (5.1.11)$$

$$\begin{aligned} \varepsilon^b(k^b) = & -2\mu - 2 \int_{-\pi}^{\pi} dk^u \cos^2 k^u a_1(\sin k^u - k^b) - \int_{Q_-^b}^{Q_+^b} d\Lambda a_2(k^b - \Lambda) \varepsilon^b(\Lambda) \\ & - \int_{Q_-^u}^{Q_+^u} dk^u \cos k^u a_1(\sin k^u - k^b) \varepsilon^u(k^u). \end{aligned} \quad (5.1.12)$$

where the same as before we use $\varepsilon^\gamma(k^\gamma)$ ($\gamma = u, b$) to denote $\kappa(k)$ and $\varepsilon'_1(\Lambda)$.

The conformal invariance of a 1D many-body system at $T = 0$ provides a universality class of criticality in terms of the central charge C of the underlying Virasoro algebra. Indeed, the dimensionless central charge classifies the finite-size scaling form of energies in low-lying excitations. In particular, the $C = 1$ universality class gives rise to a systematic calculation of the critical exponents which govern the power-law decay of correlation functions in long-distance [74–76]. With the help of root densities presented in eqs. (5.1.9) and (5.1.10) and the TBA equations (5.1.11) and (5.1.12), we derive the finite-size correction to the ground state energy in the form

$$\frac{\Delta E}{L} = - \sum_{\gamma=u,b} \frac{C\pi}{6L^2} v^\gamma, \quad (5.1.13)$$

where $C = 1$ is the central charge for both branches of excitations. In the above equations $v^{u,b}$ are the sound velocities of the unpaired fermions and bound pairs. The sound velocities are defined by $v^\gamma = \pm \frac{d\varepsilon^\gamma(k^\gamma)}{dp^\gamma(k^\gamma)} \Big|_{k^\gamma=\pm Q_\gamma} = \pm \frac{d\varepsilon^\gamma/dk^\gamma}{2\pi\rho^\gamma(k^\gamma)} \Big|_{k^\gamma=\pm Q_\gamma}$ ($\gamma = u, b$). Here we denoted $\pm Q_\gamma$ as the Fermi points of corresponding single fermions and bound pairs and the momenta $p^\gamma(k^\gamma) = \lim_{L \rightarrow \infty} y_L^\gamma(k^\gamma)$.

5.2 Low-Lying Excitations and Dressed Charge Matrix

In order to obtain the conformal dimensions and the critical exponents, one needs to calculate the finite-size corrections to the low-lying excitations. The 1D attractive Hubbard model has two branches of excitations in terms of the unpaired fermions and bound pairs. Here we follow the scheme established in Refs. [8, 37, 38, 40]. In general, the low-lying excitations can be realized by combination of three types of elementary excitations, all of which involve the distortion of the Fermi points due to the transformation or variation of the Fermi points. Such changes can be characterized by the changes in the quantum numbers given in eq. (5.1.4).

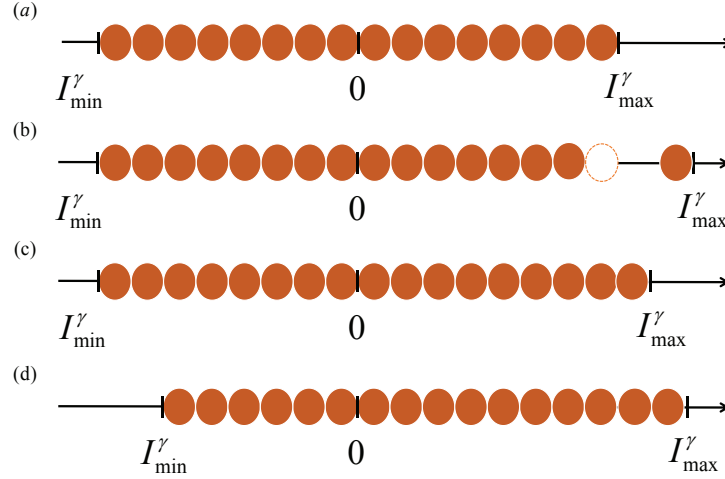


Figure 5.1: A sketch of distributions for the quantum number I_j^γ ($\gamma = u, b$) in ground state and excited states. There are three types of elementary excitations: (a) the symmetric distribution for the ground state; (b) a particle-hole excitation near the right Fermi point, i.e. the Type I excitation; (c) one more particle is added to the right Fermi point, i.e. Type II excitation; (d) two particles near the left Fermi point are moved to the right Fermi point, i.e., Type III excitation. One may find that all the possible vacancies for quantum numbers in Type II and Type III excitations are occupied by particles, which are different from the Type I excitation.

Type I excitations move particles inside the Fermi sea to location j outside the Fermi sea, known as a particle-hole excitation. The lowest particle-hole excitation is described by the change of quantum numbers I_j^γ close to I_\pm^γ ($\gamma = u, b$), where $I_+^\gamma = I_{max}^\gamma + \frac{1}{2}$ and $I_-^\gamma = I_{min}^\gamma - \frac{1}{2}$. The changes of the total momentum and energy are given by

$$\Delta P = \frac{2\pi}{L} \sum_{\gamma=u,b} (N_+^\gamma - N_-^\gamma) \mod 2\pi, \quad (5.2.1)$$

$$\Delta E = \sum_{\gamma=u,b} \frac{2\pi}{L} v^\gamma (N_+^\gamma + N_-^\gamma), \quad (5.2.2)$$

where $N_+^\gamma > 0$ ($N_-^\gamma > 0$) stands for the change of the quantum number for corresponding particle near the right (left) Fermi point.

Type II excitations originate from the change of particle numbers of unpaired fermions and bound pairs in the Fermi seas. It is easy to see that the particle

number is given by $N^\gamma = I_+^\gamma - I_-^\gamma$, and a Type II excitation is characterized by quantum number $\Delta N^\gamma = N_e^\gamma - N_g^\gamma$ where subscripts e and g respectively represent the excited state and ground state.

Type III excitations are caused by the backscattering process, where particles from one Fermi point move to the other one. They are characterized by quantum number $\Delta D^\gamma = \frac{I_+^\gamma - I_-^\gamma}{2}$, ($\gamma = u, b$). In light of the ‘continuous’ Fermi sea in the Type II and Type III excitations, quantum number ΔN^γ and ΔD^γ characterize the changes of the sizes of Fermi seas and the displacements of the center of Fermi seas, respectively.

The calculation of finite-size corrections for Type I excitations is straightforward. Calculations for the Type II and Type III excitations can also be carried out in a systematic way. See Appendix F for the detailed calculations. Here we summarize the results for the three types of elementary excitations. For convenience in the following calculation of the conformal dimensions, the excitations can be cast into unified finite-size scaling forms of the energy and total momentum, which read

$$\Delta E = \frac{2\pi}{L} \left[\frac{1}{4} \left(\Delta \vec{N} \right)^{Tr} \cdot \left(\hat{Z}^{-1} \right)^{Tr} \cdot \hat{S}_v \cdot \hat{Z}^{-1} \cdot \Delta \vec{N} + \left(\Delta \vec{D} \right)^{Tr} \cdot \hat{Z} \cdot \hat{S}_v \cdot \hat{Z}^{Tr} \cdot \Delta \vec{D} + v^\alpha \left(N_+^\alpha + N_-^\alpha \right) \right] + \text{h.o.c.}, \quad (5.2.3)$$

$$\Delta P = \frac{2\pi}{L} \sum_{\alpha=u,b} \Delta D^\alpha \cdot \left[N_+^\alpha - N_-^\alpha + \Delta D^\alpha \left(N^\alpha + \Delta N^\alpha \right) \right], \quad (5.2.4)$$

where we have introduced the notation

$$\Delta \vec{N} = \begin{bmatrix} \Delta N^u \\ \Delta N^b \end{bmatrix}, \quad \Delta \vec{D} = \begin{bmatrix} \Delta D^u \\ \Delta D^b \end{bmatrix}, \quad (5.2.5)$$

$$\hat{S}_v = \begin{bmatrix} v^u & 0 \\ 0 & v^b \end{bmatrix}, \quad \hat{Z} = \begin{bmatrix} Z_{uu}(k^u = Q_+^u) & Z_{ub}(k^b = Q_+^b) \\ Z_{bu}(k^u = Q_+^u) & Z_{bb}(k^b = Q_+^b) \end{bmatrix} \Big|_G. \quad (5.2.6)$$

In the above equations, the superscript Tr stands for transpose of matrix as before, and the quantum numbers ΔD^β ($\beta = u, b$) obey the relations

$$\Delta D^u \equiv \frac{\Delta N^u + \Delta N^b}{2} \mod 1, \quad \Delta D^b \equiv \frac{\Delta N^u}{2} \mod 1. \quad (5.2.7)$$

The dressed charges at the Fermi points Q_+^γ ($\gamma = u, b$) are obtained from the

elements of the dressed charge matrix, which satisfy the integral equations

$$Z_{uu}(k^u) = 1 - \int_{Q_-^b}^{Q_+^b} dk^b a_1(k^b - \sin k^u) Z_{ub}(k^b), \quad (5.2.8)$$

$$Z_{ub}(k^b) = - \int_{Q_-^u}^{Q_+^u} dk^u \cos k^u a_1(\sin k^u - k^b) Z_{uu}(k^u) - \int_{Q_-^b}^{Q_+^b} d\tilde{k}^b a_2(k^b - \tilde{k}^b) Z_{ub}(\tilde{k}^b), \quad (5.2.9)$$

$$Z_{bu}(k^u) = - \int_{Q_-^b}^{Q_+^b} dk^b a_1(k^b - \sin k^u) Z_{bb}(k^b), \quad (5.2.10)$$

$$Z_{bb}(k^b) = 1 - \int_{Q_-^u}^{Q_+^u} dk^u \cos k^u a_1(\sin k^u - k^b) Z_{bu}(k^u) - \int_{Q_-^b}^{Q_+^b} d\tilde{k}^b a_2(k^b - \tilde{k}^b) Z_{bb}(\tilde{k}^b). \quad (5.2.11)$$

Here the form of the dressed charges is quite different from that for the 1D repulsive Hubbard model [8].

In the ground state, phase V is gapped in the spin sector due to the existence of the bound pairs. However, the system becomes gapless if the magnetic field is greater than the lower critical field, at which bound pairs break. Consequently, phase IV consists of both bound pairs and unpaired fermions. The conformal invariant symmetry enables one to obtain the finite-size scaling forms of eqs. (5.1.13) and (5.2.3). In what follows we will calculate the conformal dimensions which determine the critical exponents of two-point correlation functions between primary fields $\langle \hat{\mathcal{O}}^\dagger(x, t) \hat{\mathcal{O}}(x', t') \rangle$.

We focus on the asymptotics of correlation functions in phase IV. For this phase, one expects a power-law decay of the correlation function at $T = 0$. Meanwhile, at $T > 0$, the correlation functions should decay exponentially. From the conformal field theory, the conformal dimensions can be read off as

$$2\Delta_\pm^u = \left(\hat{Z}_{uu} \cdot \Delta D^u + \hat{Z}_{bu} \cdot \Delta D^b \pm \frac{\hat{Z}_{bb} \cdot \Delta N^u - \hat{Z}_{ub} \cdot \Delta N^b}{2 \det\{\hat{Z}\}} \right)^2 + 2N_\pm^u \quad (5.2.12)$$

$$2\Delta_\pm^b = \left(\hat{Z}_{ub} \cdot \Delta D^u + \hat{Z}_{bb} \cdot \Delta D^b \pm \frac{\hat{Z}_{uu} \cdot \Delta N^b - \hat{Z}_{bu} \cdot \Delta N^u}{2 \det\{\hat{Z}\}} \right)^2 + 2N_\pm^b, \quad (5.2.13)$$

where N_\pm^α ($\alpha = u, b$) characterizes the descendent field from the primary field. It follows that the long-distance asymptotics of the two point correlation functions are given by

$$\langle \hat{\mathcal{O}}(x, t) \hat{\mathcal{O}}(0, 0) \rangle = \frac{\exp \left[-i \frac{2\pi}{L} (N^u \cdot \Delta D^u + N^b \cdot \Delta D^b) \right]}{(x - i\nu^u t)^{2\Delta_+^u} (x + i\nu^u t)^{2\Delta_-^u} (x - i\nu^b t)^{2\Delta_+^b} (x + i\nu^b t)^{2\Delta_-^b}}. \quad (5.2.14)$$

The dressed charge equations can be simplified in the low density regime, i.e., with small integral boundaries $Q_\gamma \ll 1$ ($\gamma = u, b$). Here we replace Q_\pm^γ by $\pm Q_\gamma$ in the dressed charge matrix, whose elements are calculated for the ground state. Obviously, the dressed charge equations can be separated into two sets of coupled integral equations, composed of eqs. (5.2.8) and (5.2.9), and of eqs. (5.2.10) and (5.2.11), respectively. By analysing the order of Q_γ in the dressed charge equations (5.2.8) to (5.2.11), we can further obtain asymptotic forms of these equations.

To begin with, we substitute eq. (5.2.9) into eq. (5.2.8) to give

$$Z_{uu}(k^u) \approx 1. \quad (5.2.15)$$

We further substitute this equation into eq. (5.2.9) to readily obtain

$$\begin{aligned} Z_{ub}(k^b) &\approx - \int_{-Q_u}^{Q_u} dk^u \cos k^u a_1(\sin k^u - k^b) - \int_{-Q_b}^{Q_b} d\tilde{k}^b a_2(k^b - \tilde{k}^b) Z_{ub}(\tilde{k}^b) \\ &\approx - \frac{1}{\pi} \arctan \left(\frac{\sin k^u - k^b}{|u|} \right) \Big|_{k^u=-Q_u}^{k^u=Q_u} - \int_{-Q_b}^{Q_b} d\tilde{k}^b a_2(k^b - \tilde{k}^b) Z_{ub}(\tilde{k}^b) \\ &\approx - \frac{2Q_u}{\pi|u|}. \end{aligned} \quad (5.2.16)$$

Similarly, we have

$$\begin{aligned} Z_{bu}(k^u) &\approx - \int_{-Q_b}^{Q_b} dk^b a_1(k^b - \sin k^u) \\ &\approx - \frac{1}{\pi} \arctan \left(\frac{k^b - \sin k^u}{|u|} \right) \Big|_{k^b=-Q_b}^{k^b=Q_b} \approx - \frac{2Q_b}{\pi|u|}, \end{aligned} \quad (5.2.17)$$

$$\begin{aligned} Z_{bb}(k^b) &\approx 1 - \int_{-Q_b}^{Q_b} d\tilde{k}^b a_2(k^b - \tilde{k}^b) \\ &\approx 1 - \frac{1}{\pi} \arctan \left(\frac{k^b - \tilde{k}^b}{2|u|} \right) \Big|_{\tilde{k}^b=-Q_b}^{\tilde{k}^b=Q_b} \approx 1 - \frac{Q_b}{\pi|u|}. \end{aligned} \quad (5.2.18)$$

Thus we obtain the dressed charge matrix to the leading order, namely

$$\hat{Z} \approx \begin{bmatrix} 1 & -\frac{2Q_u}{\pi|u|} \\ -\frac{2Q_b}{\pi|u|} & 1 - \frac{Q_b}{\pi|u|} \end{bmatrix}. \quad (5.2.19)$$

Equation (4.1.24) tells us the cut-off quasi-momenta in terms of particle den-

sities [66],

$$Q_u \approx \pi n_u + \frac{2\pi}{|u|} n_u n_b, \quad (5.2.20)$$

$$Q_b \approx \frac{\pi}{\beta_1} n_b + \frac{\pi}{\beta_1^2 |u|} n_b (n_b + 2n_u), \quad (5.2.21)$$

where the density $n_\gamma = N^\gamma/L$ ($\gamma = u, b$) must satisfy both conditions $n_\gamma/\beta_1 \ll 1$ and $n_\gamma \ll 1$. The lattice effect parameter β_1 has been defined in eq. (4.0.7).

Furthermore, in the strong coupling regime, we have $\beta_1 \approx 2/|u|$. Without losing generality, the approximation used here requires that n_γ ($\gamma = u, b$) is less than the order of $1/|u|$. Meanwhile the condition $|n_\gamma/\beta_1| \ll 1$ is required. For weak coupling regime, numerical calculation enables one to confirm asymptotic behaviour of the correlation functions. In the fig. 5.2, we show the numerical solution of the dressed charge equations eqs. (5.2.8) to (5.2.11).

We then substitute eqs. (5.2.20) and (5.2.21) into eq. (5.2.19). Using the leading order of n_γ ($\gamma = u, b$), we have the following form of the dressed charge

$$\hat{Z} \approx \begin{bmatrix} 1 & -\frac{2n_u}{|u|} \\ -\frac{2n_b}{|u|\beta_1} & 1 - \frac{n_b}{\beta_1|u|} \end{bmatrix}. \quad (5.2.22)$$

With the help of this dressed charge matrix, the conformal dimension given in eqs. (5.2.12) and (5.2.13) in the low-density regime can be approximated as

$$2\Delta_\pm^u \approx \left(\Delta D^\pm \pm \frac{1}{2} \Delta N^u \right)^2 + 2 \left(\Delta D^\pm \pm \frac{1}{2} \Delta N^u \right) \left(-\frac{2n_b}{|u|\beta_1} \Delta D^\pm \pm \frac{n_u}{|u|} \Delta N^b \right) + 2N_\pm^u \quad (5.2.23)$$

$$2\Delta_\pm^b \approx \left(\Delta D^\pm \pm \frac{1}{2} \Delta N^b \right)^2 + 2 \left(\Delta D^\pm \pm \frac{1}{2} \Delta N^b \right) \left\{ -\frac{2n_u}{|u|} \Delta D^\pm + \frac{n_b}{|u|\beta_1} \left[-\Delta D^\pm \pm \left(\Delta N^u + \frac{1}{2} \Delta N^b \right) \right] \right\} + 2N_\pm^b. \quad (5.2.24)$$

These results provide us with a direct calculation of the asymptotics of correlation functions.

5.3 Asymptotic Behavior of Correlation Functions at Zero Temperature

We study four types of correlation functions including the single-particle Green's function $G_\uparrow(x, t) = \langle \hat{c}_{x,\uparrow}^\dagger(t) \hat{c}_{0,\uparrow}(0) \rangle$, the charge density correlation function $G_m(x, t) = \langle \hat{n}_x^\dagger(t) \hat{n}_0^\dagger(0) \rangle$, the spin correlation function $G_z(x, t) = \langle \hat{s}_x^\dagger(t) \hat{s}_0^\dagger(0) \rangle$, and the pair cor-

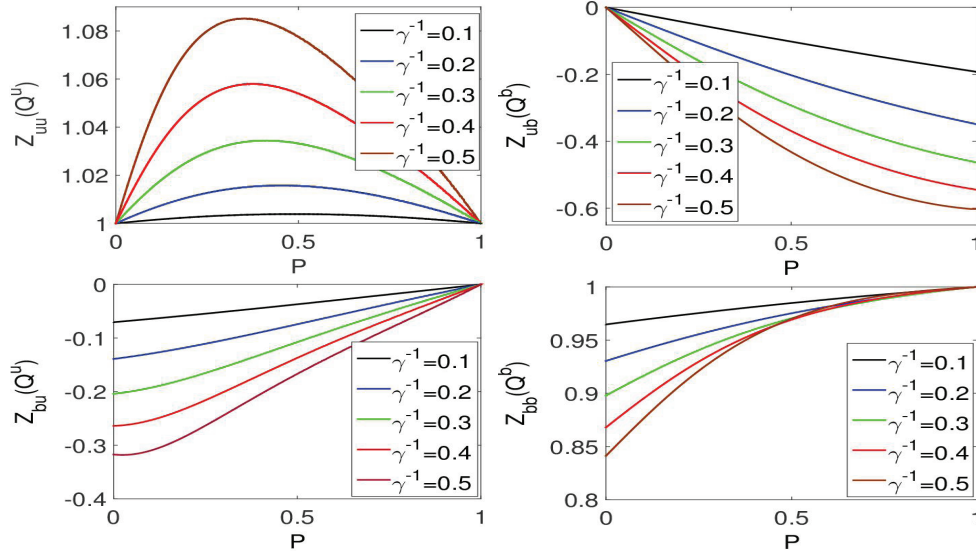


Figure 5.2: Numerical results for the dressed charges $Z_{uu}(Q_u)$, $Z_{ub}(Q_b)$, $Z_{bu}(Q_u)$, and $Z_{bb}(Q_b)$ vs polarization for different values of interaction strength. Here we have denoted $\gamma = n/|u|$. The numerical solution is obtained by solving the dressed charge equations (5.2.8-5.2.11).

relation function $G_p(x, t) = \langle \hat{c}_{x,\uparrow}^\dagger(t) \hat{c}_{x,\downarrow}^\dagger(t) \hat{c}_{0,\uparrow}(0) \hat{c}_{0,\downarrow}(0) \rangle$. Here $n(x, t) = \hat{c}_{x,\uparrow}^\dagger(t) \hat{c}_{x,\uparrow}(t) + \hat{c}_{x,\downarrow}^\dagger(t) \hat{c}_{x,\downarrow}(t)$ and $S^z(x, t) = \frac{1}{2} [\hat{c}_{x,\uparrow}^\dagger(t) \hat{c}_{x,\uparrow}(t) - \hat{c}_{x,\downarrow}^\dagger(t) \hat{c}_{x,\downarrow}(t)]$. Each of those correlation functions is accessible through choosing suitable quantum numbers ΔN^γ ($\gamma = u, b$) with respect to the low-lying excitations.

The single-particle Green's function decays exponentially if the magnetic field $B < B_{c1}$, for which the external field does not provide enough energy to break up bound pairs. However, if $B_{c1} < B < B_{c2}$ then the excess unpaired fermions appear in this gapless phase. In this regime every correlation function satisfies power-law decay [73–76, 108]. The single-particle Green's function is determined by the quantum numbers $(\Delta N^u, \Delta N^b) = (1, 0)$, which results in $(\Delta D^u, \Delta D^b) \in (\mathbb{Z} + 1/2, \mathbb{Z} + 1/2)$. Using eq. (5.2.14) the leading terms of the single-particle Green's function are

$$G_\uparrow(x, t) \approx A_{\uparrow,1} \frac{\cos(\pi(n_\uparrow - 2n_\downarrow))}{|x + i\nu^u t|^{2\theta_1} |x + i\nu^b t|^{2\theta_2}} + A_{\uparrow,2} \frac{\cos(\pi n_\uparrow)}{|x + i\nu^u t|^{2\theta_3} |x + i\nu^b t|^{2\theta_4}}, \quad (5.3.1)$$

where the critical exponents are given by

$$\theta_1 \approx 1 + \frac{2n_b}{|u|\beta_1}, \quad \theta_2 \approx \frac{1}{2} + \frac{2n_u}{|u|} - \frac{n_b}{|u|\beta_1}, \quad \theta_3 \approx 1 - \frac{2n_b}{|u|\beta_1}, \quad \theta_4 \approx \frac{1}{2} - \frac{n_u}{|u|} - \frac{n_b}{|u|\beta_1}. \quad (5.3.2)$$

The leading order term is associated with the quantum numbers $(\Delta D^u, \Delta D^b) = \pm(1/2, -1/2)$, with the next term coming from $(\Delta D^u, \Delta D^b) = \pm(1/2, 1/2)$. The coefficients $A_{\uparrow,1}$ and $A_{\uparrow,2}$ cannot be derived from the CFT approach, yet it does not impede our understanding of the long-distance asymptotic behavior of correlation functions. Here we have introduced particle densities $n_{\uparrow} = (N^u + N^b)/L$ and $n_{\downarrow} = N^b/L$ for the particles with up and down spins.

We turn to the charge density correlation function and spin correlation function, both of which are characterized by quantum numbers $(\Delta N^u, \Delta N^b) = (0, 0)$, implying $(\Delta D^u, \Delta D^b) \in (\mathbb{Z}, \mathbb{Z})$. The leading terms are expressed as

$$G_{nn} \approx n^2 + A_{nn,1} \frac{\cos(2\pi(n_{\uparrow} - n_{\downarrow})x)}{|x + i\nu^u t|^{2\theta_1}} + A_{nn,2} \frac{\cos(2\pi n_{\downarrow}x)}{|x + i\nu^b t|^{2\theta_2}} + A_{nn,3} \frac{\cos(2\pi(n_{\uparrow} - 2n_{\downarrow})x)}{|x + i\nu^u t|^{2\theta_3} |x + i\nu^b t|^{2\theta_4}}, \quad (5.3.3)$$

$$G_z \approx m_z^2 + A_{z,1} \frac{\cos(2\pi(n_{\uparrow} - n_{\downarrow})x)}{|x + i\nu^u t|^{2\theta_1}} + A_{z,2} \frac{\cos(2\pi n_{\downarrow}x)}{|x + i\nu^b t|^{2\theta_2}} + A_{z,3} \frac{\cos(2\pi(n_{\uparrow} - 2n_{\downarrow})x)}{|x + i\nu^u t|^{2\theta_3} |x + i\nu^b t|^{2\theta_4}}, \quad (5.3.4)$$

where the critical exponents are given by

$$\theta_1 \approx 2, \quad \theta_2 \approx 2 - \frac{4n_b}{|u|\beta_1}, \quad \theta_3 \approx 2 + \frac{8n_b}{|u|\beta_1}, \quad \theta_4 \approx 2 + \frac{8n_u}{|u|} - \frac{4n_b}{|u|\beta_1}. \quad (5.3.5)$$

Here the constant terms n^2 and m_z^2 originate from quantum numbers $(\Delta D^u, \Delta D^b) = (0, 0)$, while the second, third, and fourth terms come from $(\Delta D^u, \Delta D^b) = \pm(1, 0)$, $\pm(0, 1)$ and $\pm(-1, 1)$ respectively.

Last but not least we discuss the pair correlation function $G_p(x, t)$, which is described by the quantum numbers $(\Delta N^u, \Delta N^b) = (0, 1)$, allowing $(\Delta D^u, \Delta D^b) \in (\mathbb{Z} + 1/2, \mathbb{Z})$. We find that the leading terms of the pair correlation function are

$$G_p(x, t) \approx A_{p,1} \frac{\cos(\pi(n_{\uparrow} - n_{\downarrow})x)}{|x + i\nu^u \cdot t|^{2\theta_1} |x + i\nu^b \cdot t|^{2\theta_2}} + A_{p,2} \frac{\cos(\pi(n_{\uparrow} - 3n_{\downarrow})x)}{|x + i\nu^u \cdot t|^{2\theta_3} |x + i\nu^b \cdot t|^{2\theta_4}}, \quad (5.3.6)$$

with critical exponents

$$\theta_1 \approx \frac{1}{2}, \quad \theta_2 \approx \frac{1}{2} + \frac{n_b}{|u|\beta_1}, \quad \theta_3 \approx \frac{1}{2} - \frac{4n_b}{|u|\beta_1}, \quad \theta_4 \approx \frac{5}{2} - \frac{4n_u}{|u|} - \frac{3n_b}{|u|\beta_1}. \quad (5.3.7)$$

Here the first and the second terms are associated with $(\Delta D^u, \Delta D^b) = \pm(\frac{1}{2}, 0)$ and $\pm(\frac{1}{2}, 1)$, respectively.

We would like to mention that the obtained asymptotic behaviour of correlation functions reveal the important many-body correlation nature and lattice effect, see the lattice parameter-dependent critical exponents. In particular, the leading order terms of the pair correlation function and spin correlation function

reveal the spatial oscillating behavior in their long-distance asymptotics. The pair correlation function oscillates with a wave number $\Delta k = \pi(n_\uparrow - n_\downarrow)$. So does the spin correlation function with $2\Delta k$. This is for the first time to confirm the numerical finding of this nature in the literature [59–63]. The oscillation stems from the backscattering processes in the two Fermi seas, where the imbalance between the densities of spin-up and spin-down particles results in the mismatch of their Fermi surfaces. It is interesting to see that the oscillation in the spatial space in the 1D attractive Hubbard model presents the feature of the Larkin-Ovchinnikov phase predicted in [104]. We find that the oscillation terms in the spin and pair correlation functions arise from the Type III elementary excitations (backscattering process). Our theoretical result of wave number has shown good agreement with the numerics in [59], where the numerical wave number is almost $\Delta k = \pi(n_\uparrow - n_\downarrow)$. This comparison implies that the coefficient $A_{p,1}$ is much larger than $A_{p,2}$. Moreover, with respect to the coefficients in the spin correlation function, $A_{z,1}$ is much larger than $A_{z,2}$ and $A_{z,3}$ too. Finally, we would like to point out that the parameter β_1 represents the lattice effect, and thus distinguishes the Hubbard model from its continuum limit, i.e., the 1D attractive $SU(2)$ Fermi gas [66].

Applying Fourier transformation to the above correlation functions, one can derive their counterpart in momentum space [38]. For the equal-time correlation function, we have

$$g(x, t = 0^+) = \frac{\exp(ik_0 x)}{(x - i0)^{2\Delta_+} (x + i0)^{2\Delta_-}}, \quad (5.3.8)$$

where $\Delta_\pm = \Delta_\pm^u + \Delta_\pm^b$. Its Fourier transformation is

$$\tilde{g}(k \approx k_0) \sim [\text{sign}(k - k_0)]^{2s} |k - k_0|^{\mathbf{v}}. \quad (5.3.9)$$

Here the conformal spin and the exponent are given by $s = \Delta_+ - \Delta_-$ and $\mathbf{v} = 2(\Delta_+ + \Delta_-) - 1$, respectively. Consequently, their Fourier transformations of the equal-time correlation functions near the singularities $k \approx k_0$ are expressed as follows

$$\tilde{G}_\uparrow(k) \sim [\text{sign}(k - \pi(n_\uparrow - 2n_\downarrow))]^{2s_\uparrow} |k - \pi(n_\uparrow - 2n_\downarrow)|^{\mathbf{v}_\uparrow}, \quad (5.3.10)$$

$$\tilde{G}_{mn}(k) \sim [\text{sign}(k - 2\pi(n_\uparrow - n_\downarrow))]^{2s_{mn}} |k - 2\pi(n_\uparrow - n_\downarrow)|^{\mathbf{v}_{mn}}, \quad (5.3.11)$$

$$\tilde{G}_z(k) \sim [\text{sign}(k - 2\pi(n_\uparrow - n_\downarrow))]^{2s_z} |k - 2\pi(n_\uparrow - n_\downarrow)|^{\mathbf{v}_z}, \quad (5.3.12)$$

$$\tilde{G}_p(k) \sim [\text{sign}(k - \pi(n_\uparrow - n_\downarrow))]^{2s_p} |k - \pi(n_\uparrow - n_\downarrow)|^{\mathbf{v}_p}, \quad (5.3.13)$$

where

$$2s_\uparrow \approx 1, \quad \mathbf{v}_\uparrow \approx \frac{1}{2} + \frac{2n_u}{|u|} + \frac{n_b}{|u|\beta_1}, \quad (5.3.14)$$

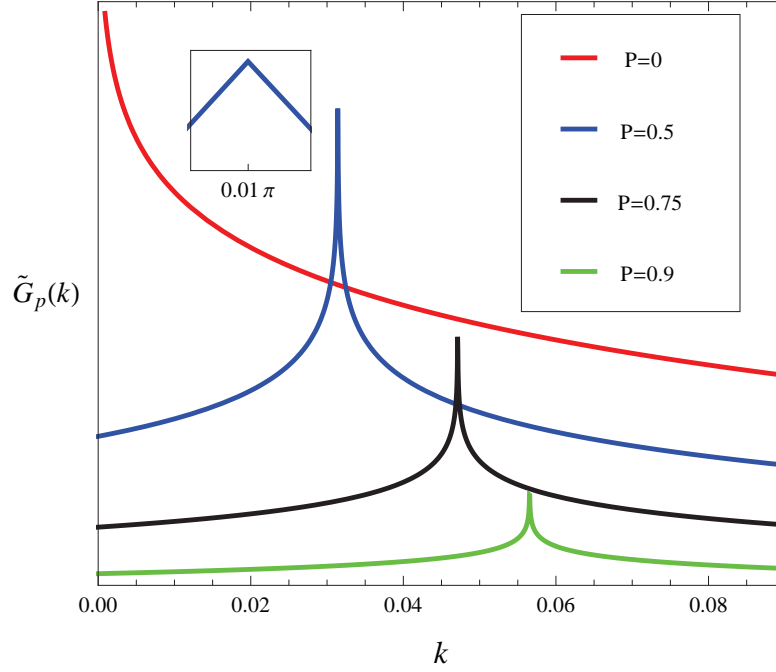


Figure 5.3: The pair correlation function in momentum space $\tilde{G}_p(k)$ vs k for different polarization $P = 0, 0.5, 0.75$, and 0.9 with interaction $u = -3$ and particle density $n = 0.02$. The inset shows the singular behavior of $\tilde{G}_p(k)$ at $k = 0.01\pi$ and $P = 0.5$. In this plot we set up natural units for the quasimomentum k .

$$2s_{nn} = 2s_z \approx 0, \quad v_{nn} = \mu_z \approx 1, \quad (5.3.15)$$

$$2s_p \approx 0, \quad v_p \approx \frac{n_b}{|u|\beta_1}. \quad (5.3.16)$$

We plot the pair correlation function in momentum space in fig. 5.3.

Notably, the correlation functions in momentum space shown in eqs. (5.3.10) to (5.3.13) validate only in the vicinity of the wave numbers k_0 , i.e. $k \approx k_0$. Figure 5.3 shows that $\tilde{G}_p(k)$ has a singularity in non-zero momentum in a partially polarized phase. This qualitatively agrees with the numerical result given in [59]. One should notice that this plot is correct only if $k \approx \pi(n_\uparrow - n_\downarrow)$, where the extrapolation is used for the purpose of a better visualization.

5.4 Conclusion

In this chapter we have investigated four types of correlation functions for the 1D attractive Hubbard model at zero temperature. The finite-size corrections to the ground state and the low-lying excitations are derived explicitly. Based on these corrections to the momentum and energy, we have applied CFT in the study of long-distance asymptotic behavior of the correlation functions. The critical exponents have been obtained explicitly in this way. In contrast to the Fermi gas, these asymptotics of correlation functions essentially depend on the parameter β_1 which represents the lattice effect. We have found that the spin and pair correlation functions have spatial oscillations with frequencies $2\pi(n_\uparrow - n_\downarrow)$ and $\pi(n_\uparrow - n_\downarrow)$, respectively. From the CFT perspective, this type of oscillation is induced by the backscattering process of unpaired fermions and bound pairs among the Fermi points. This gives a microscopic origin of the FFLO pair correlation in the 1D system. Using the Fourier transform, we have also derived the correlation functions in momentum space. The pair correlation is singular at the mismatch point $k = \pi(n_\uparrow - n_\downarrow)$, which confirms the frequency found by numerical method [59]. Meanwhile the correlation functions at zero temperature display power-law decay in the partially polarized phase IV. This suggests an analog of long range order in 1D many-body system and thus demonstrates the existence of a superconducting state.

We further point out that the dressed charge matrix in this CFT approach can be numerically resolved for arbitrary interaction strength $u < 0$ (see fig. 5.2). It follows that one can calculate the conformal dimensions and critical exponents for arbitrary interaction strength. Our result provide benchmark physics for 1D strongly correlated fermions on a lattice which may be testable in current ultracold atomic experiments [79–84].

Conclusion

With adding an on-site interaction term to the tight binding hamiltonian, the Hubbard model successfully provides a paradigm for condensed matter physics. In contrast to its simple form, this model exhibits diverse features of a many-body system, such as a Mott phase, high T_c superconductivity, quantum phase transition, FFLO phase, spin-charge separation etc. The one-dimensional Hubbard model can be exactly solved through the application of the Bethe ansatz, a thorough study of which may help us understand aspects of many-body physics in higher dimensions. Therefore the 1D attractive Hubbard model plays the central role of this thesis, where a panorama of its universal thermodynamics has been presented.

In Chapter 2, we reviewed the introduction of the Hubbard Hamiltonian, and the derivation of Lieb-Wu equations with the help of the nested Bethe ansatz. On the basis of the string hypothesis for root distribution in the attractive case, the thermodynamic Bethe ansatz (TBA) equations were obtained. We made a detailed study of the TBA equations and the root densities in the zero temperature limit. By analyzing them, we plotted the ground state phase diagram in the μ - B plane and analytically determined the phase boundaries.

In Chapter 3, we resolved the TBA equations in the strong coupling regime, and thus established the equations of state for the 1D attractive Hubbard model. The equations of state provide access to the universal thermodynamics of the system, by virtue of which we explored the behavior of thermodynamic properties. Explicit expression for the thermodynamic properties were derived, which agree well with the numerics. The scaling form of various thermodynamic quantities were obtained, from which one can easily read off the dynamical exponent $z = 2$ and correlation critical exponent $\nu = 1/2$.

In Chapter 4, we made an alternative description for the thermodynamics of FFLO-like phase in terms of an additivity rule. This simple rule reveals the free-particle nature of two Fermi liquids in the partially polarized phase IV, due to the common feature of the Luttinger liquid theory and the Fermi liquid theory — the distortion of Fermi surfaces or points. Furthermore we showed that the Wilson

ratio provides an ideal parameter to map out the various phase boundaries and to characterize the free-Fermi liquid nature of the FLLO-like phase. The Luttinger parameters were obtained too, which together with the additivity rule and Wilson ratio turns out to be the same as the case of the 1D strongly attractive $SU(2)$ Fermi gas under a lattice-gas map. This map is decided by the on-site interaction and particle density ($u \rightarrow 0$ and $n/|u| \ll 1$).

In Chapter 5, we studied the long-distance asymptotic behavior of various correlation functions in the partially polarized phase IV through the conformal field theory approach. We found the oscillating behavior of these correlation functions with spatial power-law decay. Especially, the pair (spin) correlation function oscillates with a frequency Δk_F ($2\Delta k_F$). Here $\Delta k_F = \pi(n_\uparrow - n_\downarrow)$ is the mismatch in the Fermi surfaces of spin-up and spin-down particles. Consequently, the pair correlation function in momentum space has peaks at the mismatch $k = \Delta k_F$, which has been observed in the recent numerical study of this model. These singular peaks in momentum space together with the spatial oscillation suggest an analog of the FFLO state. The parameter β_1 representing the lattice effect becomes prominent in critical exponents which determine the power-law decay of all correlation functions. We observed that the backscattering of unpaired fermions and bound pairs within their own Fermi points gives a microscopic origin of the FFLO pairing in 1D.

The results we have presented in this thesis provide a ‘rigorous’ understanding of quantum criticality, quantum liquids, and FFLO pairing correlation in the 1D attractive Hubbard model. Here for the attractive Hubbard model, we have adopted the TBA equations as an efficient tool in the study of the universal thermodynamics, and furthermore they can be applied to studying the transport properties. One can figure out the Drude weight at finite temperature by the help of finite-size correction to these TBA equations. Besides this conventional case, the thermal Drude weight could also be worked out within this framework. Meanwhile, the quantum transfer matrix is an alternative approach. The comparison between results derived from different methods will resolve the disputation on Drude weight. This future research is helpful in deepening our insight into many-body physics and can provide theoretical predictions and benchmarks for the prospective cold atom experiments.

Some Results for the Convolution of Symmetric Functions

Assuming that symmetric functions $f(x)$ and $g(x)$ are monotonous functions (MFs) of x for $x \in [0, +\infty)$, one has the following conclusions regarding their convolution $h(x) = \int_{-\infty}^{\infty} dy f(x-y)g(y)$,

- 1) $h(x)$ is a symmetric function;
- 2) $h(x)$ is a MF of x for $x \in [0, +\infty)$, where it is a monotonously increasing function (MIF) if $f(x)$ and $g(x)$ have different monotonicity, otherwise it is a monotonously decreasing function (MDF).

Here the first conclusion is easy to see through, for the second one, we give a sketch of the proof.

Proof. We consider the derivative of $h(x)$,

$$\begin{aligned}
 h'(x) &= \frac{d}{dx} \int_{-\infty}^{\infty} dy f(x-y)g(y) \\
 &= \int_{-\infty}^{\infty} dy f'(x-y)g(y) \\
 &= \int_{-\infty}^{\infty} dy f'(y)g(x-y) \\
 &= \int_0^{\infty} dy f'(y) [g(x-y) - g(x+y)], \tag{A.0.1}
 \end{aligned}$$

where we have used the fact that the derivative of a symmetric (even) function is an odd function.

Studying the integrand of eq. (A.0.1), one could find that, if $g(x)$ is a MIF (MDF) of x for $x \in [0, +\infty)$, $g(x-y) \leq g(x+y)$ ($g(x-y) \geq g(x+y)$) due to $|x-y| < x+y$ for $x, y \geq 0$; if $f(x)$ is a MIF (MDF) of x for $x \in [0, +\infty)$, $f'(x) \geq 0$ ($f'(x) \leq 0$).

Hence $h'(x)$ is non-negative if the monotonicity of $h(x)$ and $g(x)$ is different; otherwise, non-positive.

Table of Useful Integrals

B.1 Symmetric Integration

If $f(x)$ is a well-behaved function, one can derive that

$$\int_{-\pi}^{\pi} dk \cos k f(\sin k) = 2 \int_0^{\pi} dk \cos k f(\sin k) = 0. \quad (\text{B.1.1})$$

B.2 Fourier Transformation

$$\int_{-\infty}^{\infty} dx \exp(-i\omega x) a_n(x) = \exp(-n|u||\omega|). \quad (\text{B.2.1})$$

B.3 Useful Identities

Recall that

$$s(x) = \frac{1}{4|u| \cosh(\frac{\pi x}{2|u|})} = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{\exp(-i\omega x)}{2 \cosh(\omega|u|)}, \quad (\text{B.3.1})$$

$$a_n(x) = \frac{1}{2\pi} \frac{2n|u|}{(nu)^2 + x^2}, \quad (\text{B.3.2})$$

and define

$$R(x) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{\exp(i\omega x)}{1 + \exp(2|u| \cdot |\omega|)}. \quad (\text{B.3.3})$$

Thus the following identities hold

$$\int_{-\infty}^{\infty} dy s(x-y)[a_{m-1}(y) + a_{m+1}(y)] = a_m(x), \quad (\text{B.3.4})$$

$$\sum_{n=1}^{\infty} \int_{-\infty}^{\infty} dy A_{kn}^{-1}(x-y) a_n(y - \sin k) = \delta_{k,1} s(x - \sin k), \quad (\text{B.3.5})$$

$$\begin{aligned} \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} dy A_{kn}^{-1}(x-y) \left(4\text{Re} \sqrt{1 - (y - i n |u|)^2} - 2n\mu - 4n|u| \right) \\ = \delta_{k,1} \int_{-\pi}^{\pi} dk 2 \cos^2 k s(x - \sin k), \end{aligned} \quad (\text{B.3.6})$$

$$\int_{-\infty}^{\infty} d\Lambda a_1(x - \Lambda) s(\Lambda - y) = R(x - y), \quad (\text{B.3.7})$$

$$\int_{-\infty}^{\infty} d\Lambda a_1(x - \Lambda) [\delta_D(\Lambda - y) - R(\Lambda - y)] = s(x - y), \quad (\text{B.3.8})$$

$$4\text{Re} \sqrt{1 - (\Lambda - i n |u|)^2} - 4n|u| = \int_{-\pi}^{\pi} \frac{dk}{\pi} \frac{\cos^2 k 2n|u|}{(nu)^2 + (\sin k - \Lambda)^2} \quad (\text{B.3.9})$$

$$\int_{-\infty}^{\infty} dy a_n(x-y) a_m(y-z) = a_{m+n}(x-z), \quad (\text{B.3.10})$$

$$\sum_{n=0}^{\infty} (-1)^n a_{2n+1}(x) = s(x), \quad (\text{B.3.11})$$

$$\text{Re} \frac{1}{\sqrt{1 - (i \cdot n \cdot |u|)^2}} = \frac{1}{2} \int_{-\pi}^{\pi} dk a_n(\Lambda - \sin k). \quad (\text{B.3.12})$$

B.4 Integrals Involving Bessel Functions

$$J_n(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta \exp(iz \sin \theta - in\theta), \quad (\text{B.4.1})$$

$$\int_{-\pi}^{\pi} dk \cos^2 k \exp(i\omega \sin k) = \frac{2\pi J_1(\omega)}{\omega}. \quad (\text{B.4.2})$$

Properties of the Dressed Energies at $T = 0$

In this Appendix we prove that $\kappa(k)$ and $\epsilon'_1(\Lambda)$ in the zero temperature limit are symmetric functions and monotonously increase with $k > 0$ and $\Lambda > 0$ respectively. The proof is loosely analogous to that for the case of the repulsive Hubbard model in [8].

To begin with, we rewrite $\epsilon'_1(\Lambda)$ as

$$\begin{aligned} \epsilon'_1(\Lambda) = & -\mu + \int_{-\infty}^{\infty} d\Lambda' R(\Lambda - \Lambda') \epsilon'^+_1(\Lambda') - 2 \int_{-\pi}^{\pi} dk \cos^2 k s(\Lambda - \sin k) \\ & - \int_{-\pi}^{\pi} dk \cos k s(\Lambda - \sin k) \kappa^-(k), \end{aligned} \quad (\text{C.0.1})$$

which is derived by Fourier transformation. To demonstrate this, we use an alternative expression of eq. (2.3.97),

$$\begin{aligned} \epsilon'_1(\Lambda) = & -2\mu - 2 \int_{-\pi}^{\pi} dk \cos^2 k a_1(\sin k - \Lambda) - \int_{-\pi}^{\pi} dk \cos k a_1(\sin k - \Lambda) \kappa^-(k) \\ & - \int_{-\infty}^{\infty} d\Lambda' a_2(\Lambda - \Lambda') \epsilon'^-_1(\Lambda). \end{aligned} \quad (\text{C.0.2})$$

Carrying out Fourier transformation onto eq. (D.0.4) gives us,

$$\begin{aligned} \tilde{\epsilon}'_1(\omega) = & -2\mu \cdot 2\pi\delta_D(\omega) - 2 \int_{-\infty}^{\infty} d\Lambda \exp(-i\omega\Lambda) \int_{-\pi}^{\pi} dk \cos^2 k a_1(\sin k - \Lambda) \\ & - \int_{-\infty}^{\infty} d\Lambda \exp(-i\omega\Lambda) \int_{-\pi}^{\pi} dk \cos k a_1(\sin k - \Lambda) \kappa^-(k) \\ & - \tilde{a}_2(\omega)(\tilde{\epsilon}'_1(\omega) - \tilde{\epsilon}'^{+}_1(\omega)), \end{aligned} \quad (\text{C.0.3})$$

where the Fourier transformation is set as

$$\tilde{f}(\omega) = \int_{-\infty}^{\infty} dt \exp(-i\omega t) f(t), \quad (\text{C.0.4})$$

and thus the convolution theorem here we used is expressed as

$$\widetilde{f * g} = \tilde{f} \cdot \tilde{g}. \quad (\text{C.0.5})$$

After some algebraic manipulations, we obtain that

$$\begin{aligned} \tilde{\epsilon}'_1(\omega) = & -\frac{2\mu \cdot 2\pi\delta_D(\omega)}{1 + \tilde{a}_2(\omega)} + \frac{\tilde{a}_2(\omega)}{1 + \tilde{a}_2(\omega)} \cdot \tilde{\epsilon}'_1{}^+(\omega) \\ & - \frac{2}{1 + \tilde{a}_2(\omega)} \int_{-\infty}^{\infty} d\Lambda \exp(-i\omega\Lambda) \int_{-\pi}^{\pi} dk \cos^2 k a_1(\sin k - \Lambda) \\ & - \frac{1}{1 + \tilde{a}_2(\omega)} \int_{-\infty}^{\infty} d\Lambda \exp(-i\omega\Lambda) \int_{-\pi}^{\pi} dk \cos k a_1(\sin k - \Lambda) \kappa^-(k), \end{aligned} \quad (\text{C.0.6})$$

then applying inverse of the Fourier transformation gives

$$\begin{aligned} \epsilon'_1(\Lambda) = & -\mu + \int_{-\infty}^{\infty} d\omega \exp(i\omega\Lambda) \frac{1}{1 + \exp(2|u| \cdot |\omega|)} \tilde{\epsilon}'_1{}^+(\omega) \\ & - \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{2 \exp(i\omega\Lambda)}{1 + \tilde{a}_2(\omega)} \int_{-\infty}^{\infty} d\Lambda' \exp(-i\omega\Lambda') \int_{-\pi}^{\pi} dk \cos^2 k a_1(\sin k - \Lambda') \\ & - \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{\exp(i\omega\Lambda)}{1 + \tilde{a}_2(\omega)} \int_{-\infty}^{\infty} d\Lambda' \exp(-i\omega\Lambda') \int_{-\pi}^{\pi} dk \cos k a_1(\sin k - \Lambda') \kappa^-(k) \\ = & -\mu + \int_{-\infty}^{\infty} d\Lambda' R(\Lambda - \Lambda') \epsilon'_1{}^+(\Lambda') \\ & - 2 \int_{-\infty}^{\infty} d\Lambda' \int_{-\pi}^{\pi} dk \cos^2 k a_1(\sin k - \Lambda') \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \left\{ \exp[i\omega(\Lambda - \Lambda')] - \frac{\exp[i\omega(\Lambda - \Lambda')]}{1 + \exp(2|u| \cdot |\omega|)} \right\} \\ & - \int_{-\infty}^{\infty} d\Lambda' \int_{-\pi}^{\pi} dk \cos^2 k a_1(\sin k - \Lambda') \kappa^-(k) \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \left\{ \exp[i\omega(\Lambda - \Lambda')] - \frac{\exp[i\omega(\Lambda - \Lambda')]}{1 + \exp(2|u| \cdot |\omega|)} \right\} \\ = & -\mu + \int_{-\infty}^{\infty} d\Lambda' R(\Lambda - \Lambda') \epsilon'_1{}^+(\Lambda') \\ & - 2 \int_{-\infty}^{\infty} d\Lambda' \int_{-\pi}^{\pi} dk \cos^2 k a_1(\sin k - \Lambda') [\delta_D(\Lambda - \Lambda') - R(\Lambda - \Lambda')] \\ & - \int_{-\infty}^{\infty} d\Lambda' \int_{-\pi}^{\pi} dk \cos^2 k a_1(\sin k - \Lambda') \kappa^-(k) [\delta_D(\Lambda - \Lambda') - R(\Lambda - \Lambda')] \\ = & -\mu + \int_{-\infty}^{\infty} d\Lambda' R(\Lambda - \Lambda') \epsilon'_1{}^+(\Lambda) - 2 \int_{-\pi}^{\pi} dk \cos^2 k s(\Lambda - \sin k) \\ & - \int_{-\pi}^{\pi} dk \cos k s(\Lambda - \sin k) \kappa^-(k), \end{aligned} \quad (\text{C.0.7})$$

which is exactly eq. (C.0.1). Here $R(x)$ is defined as eq. (B.3.3), and in the derivation we have used eqs. (B.2.1) and (B.3.8) and the convolution theorem.

In addition, by the virtue of eq. (B.3.7), the function $R(x)$ is a symmetric function and MDF of x for $x \in [0, +\infty)$, which is an application of the second

conclusion in Appendix A.

The coupled integral equations eqs. (2.3.96) and (C.0.1) could be solved by a double iteration, with result

$$\kappa(k) = \lim_{n \rightarrow \infty} \kappa^{(n)}(k), \quad \varepsilon'_1(\Lambda) = \lim_{n \rightarrow \infty} \varepsilon_1'^{(0)}(\Lambda), \quad (\text{C.0.8})$$

$$\varepsilon_1'^{(n)} = \lim_{n \rightarrow \infty} \varepsilon_1'^{(n,m)}(\Lambda), \quad (\text{C.0.9})$$

$$\kappa^{(1)}(k) = -2 \cos k - \mu - 2u - B, \quad (\text{C.0.10})$$

$$\varepsilon_1'^{(n,1)}(\Lambda) = -2\mu, \quad (\text{C.0.11})$$

$$\begin{aligned} \varepsilon_1'^{(n,m+1)}(\Lambda) = & -\mu + \int_{-\infty}^{\infty} d\Lambda' R(\Lambda - \Lambda') \varepsilon_1'^{(n,m)+}(\Lambda') - 2 \int_{-\pi}^{\pi} dk \cos^2 k s(\Lambda - \sin k) \\ & - \int_{-\pi}^{\pi} dk \cos k s(\Lambda - \sin k) \kappa^{(n)-}(k), \end{aligned} \quad (\text{C.0.12})$$

$$\kappa^{(n+1)}(k) = \kappa^{(1)}(k) - \int_{-\infty}^{\infty} d\Lambda a_1(\sin k - \Lambda) \varepsilon_1'^{(n)-}(\Lambda). \quad (\text{C.0.13})$$

This proof is realized by the following 6 lemmas: Lemma 1 establishes the symmetric property of $\kappa^{(n)}(k)$ and $\varepsilon_1^{(n)}(\Lambda)$; Lemma 2-4 establish the existence of limits that $\kappa(k) = \lim_{n \rightarrow \infty} \kappa^{(n)}(k)$ and $\varepsilon_1^{(n)}(\Lambda) = \lim_{n \rightarrow \infty} \varepsilon_1^{(n)}(\Lambda)$; at last Lemma 5 and 6 establish the monotonicity properties of $\kappa(k)$ and $\varepsilon_1'(\Lambda)$ for their arguments in the corresponding intervals $[0, \pi]$ and $[0, +\infty)$, respectively.

Lemma 1.

$$\varepsilon_1'^{(n)}(\Lambda) = \varepsilon_1'^{(n)}(-\Lambda), \quad \kappa^{(n)}(k) = \kappa^{(n)}(-k). \quad (\text{C.0.14})$$

Proof. It is easy to prove this by induction.

Lemma 2.

$$0 \geq \int_{-\pi}^{\pi} dk \cos k s(\Lambda - \sin k) \kappa^{(n)-}(k) \quad (\text{C.0.15})$$

$$\geq \int_{-\pi}^{\pi} dk \cos k s(\Lambda - \sin k) \kappa^{(n)}(k) \quad (\text{C.0.16})$$

$$\geq -2 \int_{-\pi}^{\pi} dk \cos^2 k s(\Lambda - \sin k). \quad (\text{C.0.17})$$

Proof. We start from the first inequality. By the help of integral variable change, it is easy to obtain that

$$\int_{-\pi}^{\pi} dk \cos k s(\Lambda - \sin k) \kappa^{(n)-}(k)$$

$$\begin{aligned}
&= \left(\int_{-\pi}^{-\pi/2} + \int_{-\pi/2}^0 \right) dk \cos k s(\Lambda - \sin k) \kappa^{(n)-}(k) + \left(\int_0^{\pi/2} + \int_{\pi/2}^{\pi} \right) dk \cos k s(\Lambda - \sin k) \kappa^{(n)-}(k) \\
&= \int_0^{\pi/2} dk' \cos(k' - \pi) s(\Lambda - \sin(k' - \pi)) \kappa^{(n)-}(k' - \pi) \\
&\quad + \int_{-\pi/2}^0 dk \cos k s(\Lambda - \sin k) \kappa^{(n)-}(k) + \int_0^{\pi/2} dk \cos k s(\Lambda - \sin k) \kappa^{(n)-}(k) \\
&\quad + \int_{-\pi/2}^0 dk' \cos(k' + \pi) s(\Lambda - \sin(k' + \pi)) \kappa^{(n)-}(k' + \pi) \\
&= - \int_0^{\pi/2} dk' \cos k' s(\Lambda + \sin k') \kappa^{(n)-}(k' - \pi) + \int_{-\pi/2}^0 dk \cos k s(\Lambda - \sin k) \kappa^{(n)-}(k) \\
&\quad + \int_0^{\pi/2} dk \cos k s(\Lambda - \sin k) \kappa^{(n)-}(k) - \int_{-\pi/2}^0 dk' \cos k' s(\Lambda + \sin k') \kappa^{(n)-}(k' + \pi) \\
&= - \int_{-\pi/2}^0 dk \cos k s(\Lambda - \sin k) \kappa^{(n)-}(-k - \pi) + \int_{-\pi/2}^0 dk \cos k s(\Lambda - \sin k) \kappa^{(n)-}(k) \\
&\quad + \int_0^{\pi/2} dk \cos k s(\Lambda - \sin k) \kappa^{(n)-}(k) - \int_0^{\pi/2} dk \cos k s(\Lambda - \sin k) \kappa^{(n)-}(-k + \pi) \\
&= \int_0^{\pi/2} dk \cos k s(\Lambda - \sin k) [\kappa^{(n)-}(k) - \kappa^{(n)-}(\pi - k)] \\
&\quad + \int_{-\pi/2}^0 dk \cos k s(\Lambda - \sin k) [\kappa^{(n)-}(k) - \kappa^{(n)-}(-\pi - k)]. \tag{C.0.18}
\end{aligned}$$

From eq. (C.0.13), the definition of $\kappa^{(n)}(k)$, one knows that

$$\begin{aligned}
\kappa^{(n)}(\pi - k) - \kappa^{(n)}(k) &= 4 \cos k \geq 0, \text{ for } k \in \left[0, \frac{\pi}{2}\right], \\
\kappa^{(n)}(-\pi - k) - \kappa^{(n)}(k) &= 4 \cos k \geq 0, \text{ for } k \in \left[-\frac{\pi}{2}, 0\right], \tag{C.0.19}
\end{aligned}$$

which imply that

$$\begin{aligned}
\kappa^{(n)\pm}(\pi - k) &\geq \kappa^{(n)\pm}(k), \text{ for } k \in \left[0, \frac{\pi}{2}\right], \\
\kappa^{(n)\pm}(-\pi - k) &\geq \kappa^{(n)\pm}(k), \text{ for } k \in \left[-\frac{\pi}{2}, 0\right]. \tag{C.0.20}
\end{aligned}$$

Therefore the rhs of eq. (C.0.18) is negative and the first inequality is proved. Analogously, one could prove that

$$0 \geq \int_{-\pi}^{\pi} dk \cos k s(\Lambda - \sin k) \kappa^{(n)+}(k), \tag{C.0.21}$$

which together with the first inequality prove the second one.

The third inequality is obtained by substituting eq. (C.0.13) into the rhs of

eq. (C.0.16) and using eq. (B.1.1).

Lemma 3. The limit $\epsilon_1'^{(n)} = \lim_{m \rightarrow \infty} \epsilon_1'^{(n,m)}(\Lambda)$ exists.

Proof. Using induction, we prove that

$$\epsilon_1'^{(n,m+1)}(\Lambda) \leq \epsilon_1'^{(n,m)}(\Lambda). \quad (\text{C.0.22})$$

Induction start: In light of $\epsilon_1'^{(n,1)}(\Lambda) = \epsilon_1'^{(n,1)+}(\Lambda) = -2\mu$, we derive that

$$\begin{aligned} & \epsilon_1'^{(n,2)}(\Lambda) - \epsilon_1'^{(n,1)}(\Lambda) \\ &= -2 \int_{-\pi}^{\pi} dk \cos^2 k s(\Lambda - \sin k) - \int_{-\pi}^{\pi} dk \cos k s(\Lambda - \sin k) \kappa^{(n)-}(k) \end{aligned} \quad (\text{C.0.23})$$

$$\leq 0, \quad (\text{C.0.24})$$

where the inequality is exactly what we have proved in Lemma 2. eq. (C.0.17).

Induction step: Assuming that eq. (C.0.22) holds for $m \leq k$, we could derive that

$$\begin{aligned} & \epsilon_1'^{(n,k+1)}(\Lambda) - \epsilon_1'^{(n,k)}(\Lambda) \\ &= \int_{-\infty}^{\infty} d\Lambda' R(\Lambda - \Lambda') \left[\epsilon_1'^{(n,k)+}(\Lambda') - \epsilon_1'^{(n,k-1)+}(\Lambda') \right] \end{aligned} \quad (\text{C.0.25})$$

$$\leq 0. \quad (\text{C.0.26})$$

By now the induction is completed, which proves that $\epsilon_1'^{(n,m)}(\Lambda)$ decreases as m increases.

Meanwhile, based on eq. (C.0.12) and eq. (C.0.15), one could obtain that

$$\epsilon_1'^{(n,m)}(\Lambda) \geq -2 \int_{-\pi}^{\pi} dk \cos^2 k s(\Lambda - \sin k). \quad (\text{C.0.27})$$

The proof is completed.

Lemma 4.

$$\epsilon_1'^{(n,m)}(\Lambda) \geq \epsilon_1'^{(n+1,m+1)}(\Lambda), \quad (\text{C.0.28})$$

$$\kappa^{(n)}(k) \leq \kappa^{(n+1)}(k), \quad (\text{C.0.29})$$

$$\epsilon_1'^{(n)}(\Lambda) \geq \epsilon_1'^{(n+1)}(\Lambda). \quad (\text{C.0.30})$$

Proof. These three inequalities could be proved by mathematical induction in n .

Induction start: Through eq. (C.0.13) the definition of $\kappa^{(n)}(k)$, it is straightfor-

ward to see that

$$\kappa^{(1)}(k) \leq \kappa^{(2)}(k). \quad (\text{C.0.31})$$

Then we try to prove $\varepsilon_1'^{(1,m)}(\Lambda) \geq \varepsilon_1'^{(2,m)}(\Lambda)$, which is realized by mathematical induction in m .

The case where $m = 1$ is trivial.

The case where $m = 2$ is a little complicated,

$$\begin{aligned} & \varepsilon_1'^{(1,2)}(\Lambda) - \varepsilon_1'^{(2,2)}(\Lambda) \\ &= - \int_{-\pi}^{\pi} dk \cos k s(\Lambda - \sin k) \left[\kappa^{(1)-}(k) - \kappa^{(2)-}(k) \right] \\ &= - \int_0^{\pi/2} dk \cos k [s(\Lambda - \sin k) + s(\Lambda + \sin k)] [f_1(k) - f_2(k)], \end{aligned} \quad (\text{C.0.32})$$

where $f_n(k) = \kappa^{(n)-}(k) - \kappa^{(n)-}(\pi - k)$. It is easy to see that $f_1(k) - f_2(k)$ is the key. In proof of Lemma 2, we have known that $\kappa^{(n)}(\pi - k) - \kappa^{(n)}(k) = 4 \cos k$ for $k \in [0, \frac{\pi}{2}]$. Then we have

$$f_n(k) = \kappa^{(n)-}(k) - \left[\kappa^{(n)}(k) + 4 \cos k \right]^- \quad (\text{C.0.33})$$

$$= \begin{cases} \kappa^{(n)-}(k) & \text{if } \kappa^{(n)}(k) + 4 \cos k \geq 0, \\ -4 \cos k & \text{if } \kappa^{(n)}(k) + 4 \cos k \leq 0, \end{cases} \quad (\text{C.0.34})$$

where $k \in [0, \frac{\pi}{2}]$.

Considering that $\kappa^{(1)}(k) \leq \kappa^{(2)}(k)$, one could derive

$$\begin{aligned} & f_1(k) - f_2(k) \quad (\text{C.0.35}) \\ &= \begin{cases} 0, & \text{if } \kappa^{(2)}(k) + 4 \cos k \leq 0, \\ \kappa^{(1)-}(k) - \kappa^{(2)-}(k), & \text{if } \kappa^{(2)}(k) + 4 \cos k \geq 0 \text{ and } \kappa^{(1)}(k) + 4 \cos k \geq 0, \\ -4 \cos k - \kappa^{(2)-}(k), & \text{if } \kappa^{(2)}(k) + 4 \cos k \geq 0 \text{ and } \kappa^{(1)}(k) + 4 \cos k \leq 0, \end{cases} \end{aligned} \quad (\text{C.0.36})$$

and hence $f_1(k) - f_2(k)$ is always negative for $k \in [0, \frac{\pi}{2}]$, which means that $\varepsilon_1'^{(1,2)}(\Lambda) \geq \varepsilon_1'^{(2,2)}(\Lambda)$.

Then it is turn to the induction step over m ,

$$\begin{aligned} & \varepsilon_1'^{(1,m)}(\Lambda) - \varepsilon_1'^{(2,m)}(\Lambda) \\ &= - \int_{-\pi}^{\pi} dk \cos k s(\Lambda - \sin k) \left[\kappa^{(1)-}(k) - \kappa^{(2)-}(k) \right] \end{aligned}$$

$$+ \int_{-\infty}^{\infty} d\Lambda' R(\Lambda - \Lambda') \left[\epsilon_1'^{(1,m-1)}(\Lambda) - \epsilon_1'^{(2,m-1)}(\Lambda) \right] \quad (\text{C.0.37})$$

$$\geq 0, \quad (\text{C.0.38})$$

where the first term on the rhs of eq. (C.0.37) is exactly $\epsilon_1'^{(1,2)}(\Lambda) - \epsilon_1'^{(2,2)}(\Lambda)$, and the second term is the induction assumption, both of which are non-negative. Hence we confirm

$$\epsilon_1'^{(1,m)}(\Lambda) \geq \epsilon_1'^{(2,m)}(\Lambda).$$

According to Lemma 5, taking limit $m \rightarrow \infty$ results in

$$\epsilon_1'^{(1)}(\Lambda) \geq \epsilon_1'^{(2)}(\Lambda).$$

Induction step: The inequality $\epsilon_1'^{(1,n-1)}(\Lambda) \geq \epsilon_1'^{(1,n)}(\Lambda)$ implies that $\kappa^{(n-1)}(k) \leq \kappa^{(n)}(k)$, where we use eq. (C.0.13), the definition of $\kappa^{(n)}(k)$.

The proof to $\epsilon_1'^{(n,m)}(\Lambda) \geq \epsilon_1'^{(n+1,m)}(\Lambda)$ is similar to eq. (C.0.38). Taking limit $m \rightarrow \infty$ yields

$$\epsilon_1'^{(n)}(\Lambda) \geq \epsilon_1'^{(n+1)}(\Lambda).$$

In Lemma 4 we have demonstrated $\kappa^{(n)}(k)$ increases and $\epsilon_1'^{(n)}(\Lambda)$ decreases with increasing n . Hereafter we confirm these two functions are bounded.

For $\epsilon_1'^{(n)}(\Lambda)$, eq. (C.0.27) and the statement of Lemma 3 tell us the existence of its inferior limit.

For $\kappa^{(n)}(k)$, due to eq. (C.0.27), it is easy to know that,

$$0 \geq \epsilon_1'^{(n,m)-}(\Lambda) \geq -2 \int_{-\pi}^{\pi} dk \cos^2 k s(\Lambda - \sin k), \quad (\text{C.0.39})$$

which indicates

$$\int_{-\infty}^{\infty} d\Lambda a_1(\sin k - \Lambda) \epsilon_1'^{(n)-}(\Lambda) \geq \int_{-\infty}^{\infty} d\Lambda a_1(\sin k - \Lambda) \left[-2 \int_{-\pi}^{\pi} dk \cos^2 k s(\Lambda - \sin k) \right]. \quad (\text{C.0.40})$$

Substitute last equality into eq. (C.0.13) gives

$$\kappa^{(n)}(k) \leq \kappa^{(1)}(k) - \int_{-\infty}^{\infty} d\Lambda a_1(\sin k - \Lambda) \left[-2 \int_{-\pi}^{\pi} dk \cos^2 k s(\Lambda - \sin k) \right]. \quad (\text{C.0.41})$$

Hence there is a superior limit for $\kappa^{(n)}(k)$, which means that $\lim_{n \rightarrow \infty} \kappa^{(n)}(k)$ exists.

Lemma 5.

- a) $\epsilon_1'^{(n,m)}(\Lambda)$ is a monotonously increasing function (MIF) of Λ for $\Lambda \in [0, +\infty)$;
 b) $\kappa^{(n)}(k)$ is a monotonously increasing function (MIF) of k for $k \in [0, \frac{\pi}{2}]$.

Proof. To begin with, we rewrite the recursion for $\epsilon_1'^{(n,m+1)}(\Lambda)$ in eq. (C.0.12) as

$$\begin{aligned}
 \epsilon_1'^{(n,m+1)}(\Lambda) &= -\mu + \int_{-\infty}^{\infty} d\Lambda' R(\Lambda - \Lambda') \epsilon_1'^{(n,m)+}(\Lambda') \\
 &\quad - 2 \int_{-\pi}^{\pi} dk \cos^2 k s(\Lambda - \sin k) - \int_{-\pi}^{\pi} dk \cos k s(\Lambda - \sin k) \kappa^{(n)-}(k), \\
 &= -\mu + \int_{-\infty}^{\infty} d\Lambda' R(\Lambda - \Lambda') \epsilon_1'^{(n,m)+}(\Lambda') \\
 &\quad + \int_{-\pi}^{\pi} dk \cos k s(\Lambda - \sin k) \left[-2 \cos k - \kappa^{(n)-}(k) \right] \\
 &= -\mu + \int_{-\infty}^{\infty} d\Lambda' R(\Lambda - \Lambda') \epsilon_1'^{(n,m)+}(\Lambda') \\
 &\quad + \int_{-\pi}^{\pi} dk \int_{-\infty}^{\infty} d\Lambda' \cos k s(\Lambda - \Lambda') \left[-2 \cos k - \kappa^{(n)-}(k) \right] \delta_D(\Lambda' - \sin k) \\
 &= -\mu + \int_{-\infty}^{\infty} d\Lambda' R(\Lambda - \Lambda') \epsilon_1'^{(n,m)+}(\Lambda') + \int_{-\infty}^{\infty} d\Lambda' s(\Lambda - \Lambda') \Phi^{(n)}(\Lambda'), \\
 &\hspace{25em} (C.0.42)
 \end{aligned}$$

where we introduce a new function

$$\begin{aligned}
 \Phi^{(n)}(\Lambda) &= \int_{-\pi}^{\pi} dk \cos k \delta_D(\Lambda - \sin k) \left[-2 \cos k - \kappa^{(n)-}(k) \right] \\
 &= \int_{-\pi/2}^0 dk \cos k \delta_D(\Lambda - \sin k) \left[-4 \cos k + \kappa^{(n)-}(-\pi - k) - \kappa^{(n)}(k) \right] \\
 &\quad + \int_0^{\pi/2} dk \cos k \delta_D(\Lambda - \sin k) \left[-4 \cos k + \kappa^{(n)-}(\pi - k) - \kappa^{(n)}(k) \right] \\
 &= \int_{-\pi/2}^{\pi/2} dk \cos k \delta_D(\Lambda - \sin k) \left[-4 \cos k + \kappa^{(n)-}(\pi - k) - \kappa^{(n)}(k) \right] \\
 &= \int_{-\pi/2}^{\pi/2} dk \cos k \delta_D(\Lambda - \sin k) \left[-4 \cos k - f_n(k) \right]. \\
 &\hspace{25em} (C.0.43)
 \end{aligned}$$

Here $f_n(x)$ has been defined in eq. (C.0.34), and we have used the relation $\kappa^{(n)-}(-\pi - k) = \kappa^{(n)-}(\pi - k)$ which is induced by the periodic property of $\kappa^{(n)-}(k)$.

More specific, one could express $\Phi^{(n)}(\Lambda)$ as

$$\Phi^{(n)}(\Lambda) = \begin{cases} -4\sqrt{1 - \Lambda^2} - \kappa^{(n)-}(z), & \text{if } |\Lambda| < 1 \text{ and } \kappa^{(n)}(z) \geq -4\sqrt{1 - \Lambda^2}, \\ 0, & \text{otherwise,} \end{cases} \hspace{2em} (C.0.44)$$

where $z = \arcsin(\Lambda)$.

Obviously, based on eq. (C.0.42) and the second conclusion in appendix A, $\epsilon_1'^{(n,m+1)}(\Lambda)$ is a MIF if $\Phi^{(n)}(\Lambda)$ and $\epsilon_1'^{(n,m)}(\Lambda)$ are both MIFs of Λ for $\Lambda \in [0, +\infty)$. On the other hand, if $|\Lambda| < 1$ and $\kappa^{(n)}(z) \geq -4\sqrt{1-\Lambda^2}$, one can rewrite $\Phi^{(n)}(\Lambda)$ as $\Phi^{(n)}(\Lambda) = -4\sqrt{1-\Lambda^2} - \kappa^{(n)}(z) + \kappa^{(n)+}(z)$. Then it is easy to see that $\Phi^{(n)}(\Lambda)$ is a MIF if $-4\sqrt{1-\Lambda^2} - \kappa^{(n)}(z)$ and $\kappa^{(n)}(k)$ are both MIFs of their arguments, where we have considered the fact that $z = \arcsin(\Lambda)$ is a MIF. Hereafter we complete these proofs by mathematical induction in n .

Induction start: For the case of $n = 1$, $\kappa^{(1)}(k)$ is a MIF of k for $k \in [0, \frac{\pi}{2}]$, and it is easy to derive that

$$-4\sqrt{1-\Lambda^2} - \kappa^{(1)}(\arcsin(\Lambda)) = -2\sqrt{1-\Lambda^2} + \mu + 2u + B \quad (\text{C.0.45})$$

is a MIF of Λ for $\Lambda \in [0, +\infty)$, and thus in turn $\Phi^{(1)}(\Lambda)$ is a MIF of Λ for $\Lambda \in [0, +\infty)$.

In view of $\epsilon_1'^{(1,1)}(\Lambda) = -2\mu$ and $\kappa_1(k)$ being a MIF, using the recursion eq. (C.0.42), one can establish that $\epsilon_1'^{(1,m)}(\Lambda)$ is a MIF of Λ for $\Lambda \in [0, +\infty)$ by mathematical induction in m . Taking limit $m \rightarrow \infty$ gives $\epsilon_1'^{(1)}(\Lambda)$ is a MIF.

Induction step: The induction assumption is stated as follows, for any $p \leq n$, $-4\sqrt{1-\Lambda^2} - \kappa^{(p)}(z)$ and $\kappa^p(k)$ are MIFs of their arguments for $\Lambda \in [0, +\infty)$ and $k \in [0, \frac{\pi}{2}]$, respectively; which make $\Phi^{(p)}(\Lambda)$ to be a MIF for any $p \leq n$.

Furthermore, on account of $\epsilon_1'^{(n,1)}(\Lambda) = -2\mu$, one could establish that $\epsilon_1'^{(n,m)}(\Lambda)$ is a MIF by a mathematical induction in m through eq. (C.0.42). Take limit $m \rightarrow \infty$, we could confirm that $\epsilon_1'^{(n)}(\Lambda)$ is a MIF of Λ for $\Lambda \in [0, +\infty)$. Next we would complete the induction step.

It is not difficult to see that

$$\begin{aligned} & -4\sqrt{1-\Lambda^2} - \kappa^{(n+1)}(z) \\ &= -4\sqrt{1-\Lambda^2} - \kappa^{(1)}(z) + \int_{-\infty}^{\infty} d\Lambda a_1(\sin k - \Lambda) \epsilon_1'^{(n)-}(\Lambda) \end{aligned} \quad (\text{C.0.46})$$

is a MIF of Λ . The first two terms on the rhs has been discussed in eq. (C.0.45). For the third term of integral, one should note that $\epsilon_1'^{(n)}(\Lambda)$ is a MIF of Λ for $\Lambda \in [0, +\infty)$. Using the second conclusion in appendix A, one could confirm that this integral is a MIF of Λ too.

Hence one could say that $\Phi^{(n+1)}(\Lambda)$ is a MIF of Λ for $\Lambda \in [0, +\infty)$, and so do $\epsilon_1'^{(n+1,m)}(\Lambda)$ and $\epsilon_1'^{(n)}(\Lambda)$, the establishment for each of which is completely similar to what is done to $\epsilon_1'^{(1,m)}(\Lambda)$ and $\epsilon_1'^{(1)}(\Lambda)$.

At last we complete the establishment from induction assumption that $\kappa^{(n)}(k)$

is a MIF of k for $k \in [0, \frac{\pi}{2}]$ to the same statement for $\kappa^{(n+1)}(k)$.

We start from eq. (C.0.13)

$$\begin{aligned}\kappa^{(n+1)}(k) &= \kappa^{(1)}(k) - \int_{-\infty}^{\infty} d\Lambda a_1(\sin k - \Lambda) \epsilon_1'^{(n)-}(\Lambda) \\ &= \kappa^{(1)}(k) - \int_{-\infty}^{\infty} d\Lambda a_1(\sin k - \Lambda) \left(\epsilon_1'^{(n)}(\Lambda) - \epsilon_1'^{(n)+}(\Lambda) \right).\end{aligned}\quad (\text{C.0.47})$$

According to Lemma 3, taking limit $m \rightarrow \infty$ for eq. (C.0.12), we obtain that

$$\begin{aligned}\epsilon_1'^{(n)}(\Lambda) &= -\mu + \int_{-\infty}^{\infty} d\Lambda' R(\Lambda - \Lambda') \epsilon_1'^{(n)+}(\Lambda') - 2 \int_{-\pi}^{\pi} dk \cos^2 k s(\Lambda - \sin k) \\ &\quad - \int_{-\pi}^{\pi} dk \cos k s(\Lambda - \sin k) \kappa^{(n)-}(k),\end{aligned}\quad (\text{C.0.48})$$

substituting which into eq. (C.0.47) results in

$$\kappa^{(n+1)}(k) = \kappa^{(1)}(k) + \int_{-\infty}^{\infty} d\Lambda a_1(\sin k - \Lambda) \epsilon_1'^{(n)+}(\Lambda) + \mu \quad (\text{C.0.49})$$

$$\begin{aligned}& - \int_{-\infty}^{\infty} d\Lambda a_1(\sin k - \Lambda) \int_{-\infty}^{\infty} d\Lambda' R(\Lambda - \Lambda') \epsilon_1'^{(n)+}(\Lambda') \\ & + 2 \int_{-\infty}^{\infty} d\Lambda a_1(\sin k - \Lambda) \int_{-\pi}^{\pi} dk' \cos^2 k' s(\Lambda - \sin k') \\ & + \int_{-\infty}^{\infty} d\Lambda a_1(\sin k - \Lambda) \int_{-\pi}^{\pi} dk' \cos k' s(\Lambda - \sin k') \kappa^{(n)-}(k') \\ & = \kappa^{(1)}(k) + \int_{-\infty}^{\infty} d\Lambda a_1(\sin k - \Lambda) \epsilon_1'^{(n)+}(\Lambda) + \mu \\ & \quad + \int_{-\infty}^{\infty} d\Lambda a_1(\sin k - \Lambda) \int_{-\infty}^{\infty} d\Lambda' [\delta_D(\Lambda - \Lambda') - R(\Lambda - \Lambda') - \delta_D(\Lambda - \Lambda')] \epsilon_1'^{(n)+}(\Lambda') \\ & \quad + 2 \int_{-\pi}^{\pi} dk' \cos^2 k' \int_{-\infty}^{\infty} d\Lambda a_1(\sin k - \Lambda) s(\Lambda - \sin k') \\ & \quad + \int_{-\pi}^{\pi} dk' \cos k' \kappa^{(n)}(k') \int_{-\infty}^{\infty} d\Lambda a_1(\sin k - \Lambda) s(\Lambda - \sin k') \\ & = \kappa^{(1)}(k) + \int_{-\infty}^{\infty} d\Lambda s(\sin k - \Lambda) \epsilon_1'^{(n)+}(\Lambda) + \mu + 2 \int_{-\pi}^{\pi} dk' \cos^2 k' R(\sin k - \sin k') \\ & \quad + \int_{-\pi}^{\pi} dk' \cos k' R(\sin k - \sin k') \kappa^{(n)-}(k'), \\ & = \int_{-\infty}^{\infty} d\Lambda s(\sin k - \Lambda) \epsilon_1'^{(n)+}(\Lambda) + \int_{-\pi}^{\pi} dk' \cos k' R(\sin k - \sin k') \kappa^{(n)-}(k') \\ & \quad + \left\{ \kappa^{(1)}(k) + \mu + 2 \int_{-\pi}^{\pi} dk' \cos^2 k' R(\sin k - \sin k') \right\},\end{aligned}\quad (\text{C.0.51})$$

where we have used eq. (B.3.8).

For the first term on the rhs of eq. (C.0.51), we have established the statement that $\mathfrak{e}'^{(n)}(\Lambda)$ is a MIF of Λ in the interval of $[0, +\infty)$. According to the second conclusion of appendix A, we hence confirm this term is a MIF of k for $k \in [0, \frac{\pi}{2}]$.

For the second term, we make a transformation as follows

$$\begin{aligned} & \int_{-\pi}^{\pi} dk' \cos k' R(\sin k - \sin k') \mathfrak{K}^{(n)-}(k') \\ &= \int_{-\pi}^{\pi} dk' \int_{-\infty}^{\infty} d\Lambda \cos k' \delta_D(\Lambda - \sin k') \mathfrak{K}^{(n)-}(k') R(\sin k - \Lambda) \\ &= \int_{-\infty}^{\infty} d\Lambda R(\sin k - \Lambda) \Psi^{(n)}(\Lambda), \end{aligned} \quad (\text{C.0.52})$$

where we introduce a new function

$$\Psi^{(n)}(\Lambda) = \int_{-\pi}^{\pi} dk \cos k \Delta_D(\Lambda - \sin k) \mathfrak{K}^{(n)-}(k). \quad (\text{C.0.53})$$

Similar to what is done to $\Phi^{(n)}(\Lambda)$, we rewrite $\Psi^{(n)}(\Lambda)$ as

$$\begin{aligned} \Psi^{(n)}(\Lambda) &= \left(\int_{-\pi}^{-\pi/2} + \int_{-\pi/2}^0 \right) dk \cos k \delta_D(\Lambda - \sin k) \mathfrak{K}^{(n)-}(k) \\ &\quad + \left(\int_0^{\pi/2} + \int_{\pi/2}^{\pi} \right) dk \cos k \delta_D(\Lambda - \sin k) \mathfrak{K}^{(n)-}(k) \\ &= \int_0^{-\pi/2} dk' \cos k' \delta_D(\Lambda - \sin k') \mathfrak{K}^{(n)-}(-\pi - k') \\ &\quad + \int_{-\pi/2}^0 dk \cos k \delta_D(\Lambda - \sin k) \mathfrak{K}^{(n)-}(k) \\ &\quad + \int_0^{\pi/2} dk \cos k \delta_D(\Lambda - \sin k) \mathfrak{K}^{(n)-}(k) \\ &\quad + \int_{\pi/2}^{\pi} dk' \cos k' \delta_D(\Lambda - \sin k') \mathfrak{K}^{(n)-}(\pi - k') \\ &= \int_{-\pi/2}^{\pi/2} dk \cos k \delta_D(\Lambda - \sin k) \left[\mathfrak{K}^{(n)-}(k) - \mathfrak{K}^{(n)-}(\pi - k) \right], \end{aligned} \quad (\text{C.0.54})$$

which could be recast into

$$\Psi^{(n)}(\Lambda) = \begin{cases} \mathfrak{K}^{(n)-}(z), & \text{if } \mathfrak{K}^{(n)}(z) + 4\sqrt{1 - \Lambda^2} \geq 0, \\ -4\sqrt{1 - \Lambda^2}, & \text{if } \mathfrak{K}^{(n)}(z) + 4\sqrt{1 - \Lambda^2} \leq 0, \\ 0, & \text{if } |\Lambda| \geq 1, \end{cases} \quad (\text{C.0.55})$$

with $z = \arcsin \Lambda$.

Here we find that $\Psi^{(n)}(\Lambda)$ is a MIF of Λ in the interval of $[0, +\infty)$. Applying the second conclusion in appendix A convince us that this second term on the

rhs of eq. (C.0.51) is a MIF of k for $k \in [0, \frac{\pi}{2}]$.

For these terms in the braces, we make such a transformation

$$\begin{aligned}
& \kappa^{(1)}(k) + \mu + 2 \int_{-\pi}^{\pi} dk' \cos^2 k' R(\sin k - \sin k') \\
&= \kappa^{(1)}(k) + \mu + 2 \int_{-\pi}^{\pi} dk' \cos^2 k' \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{\exp[i\omega(\sin k - \sin k')]}{1 + \exp(2|u| \cdot |\omega|)} \\
&= \kappa^{(1)}(k) + \mu + 2 \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{\exp(i\omega \sin k)}{1 + \exp(2|u| \cdot |\omega|)} \int_{-\pi}^{\pi} dk' \cos^2 k' \exp(-i\omega \sin k') \\
&= \kappa^{(1)}(k) + \mu + 2 \int_{-\infty}^{\infty} \frac{d\omega}{\omega} \frac{J_1(\omega) \exp(i\omega \sin k)}{1 + \exp(2|u| \cdot |\omega|)} \\
&= -2 \cos k + 2 \int_{-\infty}^{\infty} \frac{d\omega}{\omega} \frac{J_1(\omega) \exp(i\omega \sin k)}{1 + \exp(2|u| \cdot |\omega|)} - 2u - B, \tag{C.0.56}
\end{aligned}$$

where we have used eqs. (B.3.3) and (B.4.2). This is definitely a MIF of k in the interval $[\frac{\pi}{2}, \pi]$, which is easy to understand by an expansion for the integral on the rhs if $|u| \ll 1$

$$\int_{-\infty}^{\infty} \frac{d\omega}{\omega} \frac{J_1(\omega) \exp(i\omega \sin k)}{1 + \exp(2|u| \cdot |\omega|)} = -\cos k + u + \frac{2}{|u|} \sum_{n=0}^{\infty} \frac{K_1(u_n)}{u_n} \cosh(u_n \sin k) \tag{C.0.57}$$

with $u_m = (n + \frac{1}{2}) \frac{\pi}{|u|}$.

One could also prove the non-negativity of the derivative of eq. (C.0.56) with respect to k

$$2 \sin k - 2 \int_{-\infty}^{\infty} d\omega \frac{J_1(\omega) \sin(\omega \sin k) \cos k}{1 + \exp(2|u| \cdot |\omega|)} \geq 0, \tag{C.0.58}$$

which means that the term in the braces is a MIF of k in the interval $[0, \frac{\pi}{2}]$.

By now, the establishment from $\kappa^{(n)}(k)$ is a MIF of k in the interval $[0, \frac{\pi}{2}]$ to the statement of $\kappa^{(n+1)}(k)$ is completed. Taking limit $n \rightarrow \infty$, one arrives at $\kappa(k)$ is a MIF of k for $k \in [0, \frac{\pi}{2}]$.

And one could repeat the procedure in dealing with $\epsilon_1'^{(n,m)}(\Lambda)$ and $\epsilon_1'^{(n)}(\Lambda)$, hence $\epsilon_1'^{(n+1,m)}(\Lambda)$ and $\epsilon_1'^{(n+1)}(\Lambda)$ are MIFs of Λ for $\Lambda \in [0, +\infty)$.

Taking limit $n \rightarrow \infty$ convinces us that $\epsilon_1'(\Lambda)$ is a MIF of Λ in the interval $[0, +\infty)$.

Lemma 6. $\kappa^{(n)}(k)$ is a MIF of k for $k \in [\frac{\pi}{2}, \pi]$.

Proof. For $n = 1$ case, according to eq. (C.0.10), the proof is trivial.

For $n \geq 2$ case, we write down eq. (C.0.13) here

$$\kappa^{(n+1)}(k) = \kappa^{(1)}(k) - \int_{-\infty}^{\infty} d\Lambda a_1(\sin k - \Lambda) \epsilon_1'^{(n)-}(\Lambda), \quad (\text{C.0.59})$$

where the second term on the rhs is a MIF of k in the interval $[\frac{\pi}{2}, \pi]$. This is resulted in by the corollary of Lemma 5 that $\epsilon_1'^{(n)}(\Lambda)$ is a MIF of Λ in the interval $[0, +\infty)$, with the application of the second conclusion in appendix A.

Hence, one can confirm that $\kappa^{(n)}(k)$ is a MIF of k for $k \in [\frac{\pi}{2}, \pi]$, and so does $\kappa(k)$ if we take limit $n \rightarrow \infty$.

Wiener-Hopf Method

The phase boundary between phases IV and V is determined by the conditions $\kappa(0) = 0$ and $\epsilon'_1(0) < 0$, which imply that $Q = 0$ and A is finite. As we have stated in section 2.3.2, at zero temperature the TBA equations in Phase V are simplified to

$$\kappa(k) = -2 \cos k - \mu - 2u - B - \int_{-A}^A d\Lambda a_1(\sin k - \Lambda) \epsilon'_1(\Lambda), \quad (\text{D.0.1})$$

$$\epsilon'_1(\Lambda) = -2\mu - 2 \int_{-\pi}^{\pi} dk \cos^2 k a_1(\sin k - \Lambda) - \int_{-A}^A d\Lambda' a_2(\Lambda - \Lambda') \epsilon'_1(\Lambda'). \quad (\text{D.0.2})$$

Particularly, if $A = \infty$, it follows that the chemical potential $\mu = 0$. The intersection of the phase boundary with the B -axis could be calculated exactly by the Fourier transformation

$$B_{c1} = 2|u| - 2 + 2 \int_0^\infty d\omega \frac{J_1(\omega) \exp(-|u|\omega)}{w \cosh(u\omega)}. \quad (\text{D.0.3})$$

Now we consider the more general case $A \gg 1$, for which the phase boundary can be resolved using the Wiener-Hopf method. By applying Fourier transformation on eq. (D.0.2) and after some algebraic manipulations, we have

$$\epsilon'_1(\Lambda) = -\mu - \int_{-\infty}^{\infty} d\omega \frac{J_1(\omega)}{\omega \cosh(u\omega)} \exp(i\omega\Lambda) + \int_0^\infty d\Lambda' \epsilon'_1(\Lambda' + A) [R(\Lambda - \Lambda' - A) + R(\Lambda + \Lambda' + A)], \quad (\text{D.0.4})$$

where the function $R(x)$ is defined in eq. (B.3.3).

Substituting $y(\Lambda) = \epsilon'_1(\Lambda + A)$ and expanding $y(\Lambda) = \sum_{n=0}^{\infty} y_n(\Lambda)$ in terms of powers of Λ in eq. (D.0.4), the result can be separated into a series of Wiener-Hopf integral equations in terms of the functions $y_n(\Lambda)$, namely

$$y_n(\Lambda) = g_n(\Lambda) + \int_0^\infty d\Lambda' R(\Lambda - \Lambda') y_n(\Lambda'). \quad (\text{D.0.5})$$

Here we denote the driving terms

$$\begin{aligned} g_0(\Lambda) &= -\mu - \int_{-\infty}^{\infty} d\omega \frac{J_1(\omega) e^{i\omega(\Lambda+A)}}{\omega \cosh(u\omega)}, \\ g_n(\Lambda) &= \int_0^{\infty} d\Lambda' R(\Lambda + \Lambda' + 2A) y_{n-1}(\Lambda'). \end{aligned} \quad (\text{D.0.6})$$

To solve these integral equations for $y_n(\Lambda)$, we begin by defining

$$\tilde{y}_n^{\pm}(\omega) = \int_{-\infty}^{\infty} d\Lambda \theta_H(\pm\Lambda) y_n(\Lambda) e^{i\omega\Lambda},$$

where $\tilde{y}_n^+(\omega)$ ($\tilde{y}_n(\omega)$) is an analytic function in the upper (lower) half-plane. It is obvious that the Fourier transformation of $y_n(x)$ satisfies the relation $\tilde{y}_n(\omega) = \tilde{y}_n^+(\omega) + \tilde{y}_n^-(\omega)$.

From eq. (D.0.5) it follows that

$$\tilde{y}_n^+(\omega) \frac{1}{1 + \exp(-2|u||\omega|)} + \tilde{y}_n^-(\omega) = \tilde{g}_n(\omega), \quad (\text{D.0.7})$$

by applying Fourier transformation. We further decompose the denominator $1 + \exp(-2|u||\omega|)$ into a product of two pieces,

$$1 + \exp(-2|u||\omega|) = G^+(\omega)G^-(\omega), \quad (\text{D.0.8})$$

where $G^+(\omega)$ ($G^-(\omega)$) is an analytic function in the upper (lower) half-plane. Then substituting this last equation into eq. (D.0.7) results in the form

$$\tilde{y}_n^+(\omega)/G^+(\omega) + G^-(\omega)\tilde{y}_n^-(\omega) = G^-(\omega)\tilde{g}_n(\omega). \quad (\text{D.0.9})$$

Furthermore, we decompose $G^-(\omega)\tilde{g}_n(\omega)$ into a sum of two pieces,

$$G^-(\omega)\tilde{g}_n(\omega) = Q_n^+(\omega) + Q_n^-(\omega), \quad (\text{D.0.10})$$

where similarly $Q_n^+(\omega)$ ($Q_n^-(\omega)$) is an analytic function in the upper (lower) half-plane. Then substitution of this last equation into eq. (D.0.9) gives

$$\tilde{y}_n^+(\omega) = G^+(\omega)Q_n^+(\omega), \quad (\text{D.0.11})$$

$$\tilde{y}_n^-(\omega) = Q_n^-(\omega)/G^-(\omega). \quad (\text{D.0.12})$$

In this way we can work out the Fourier transformation of $y_0(\Lambda)$ and $y_n(\Lambda)$ itself.

To this end, recalling (D.0.10), we firstly decompose $1 + \exp(-2|u||\omega|)$ as

$$G^+(\omega) = G^-(-\omega) = \frac{\sqrt{2\pi}}{\Gamma(\frac{1}{2} - \frac{i|u|\omega}{\pi})} \left(-\frac{i|u|\omega}{\pi} \right)^{-\frac{i|u|\omega}{\pi}} \exp\left(\frac{i|u|\omega}{\pi}\right), \quad (\text{D.0.13})$$

where we should note that $\lim_{\omega \rightarrow \infty} G^\pm(\omega) = 1$, along with the special values $G^\pm(0) = \sqrt{2}$ and $G^\pm\left(\pm \frac{i\pi}{2|u|}\right) = \sqrt{\pi/e}$ of these functions.

The decomposition for $G^-(\omega)\tilde{g}_n(\omega)$ in general is subtle, however the leading case $G^-(\omega)\tilde{g}_0(\omega)$ is accessible. We start analysis from the Fourier transformation of $g_0(\Lambda)$,

$$\tilde{g}_0(\omega) = -\mu 2\pi \delta_D(\omega) - \frac{2\pi J_1(\omega) \exp(-i\omega A)}{\omega \cosh(u\omega)}, \quad (\text{D.0.14})$$

where on the rhs the δ_D function could be decomposed as

$$2\pi \delta_D(\omega) = i \left(\frac{1}{\omega + i\varepsilon} - \frac{1}{\omega - i\varepsilon} \right) \quad (\varepsilon \rightarrow +0). \quad (\text{D.0.15})$$

The second term on the rhs is a meromorphic function of ω with poles located at

$$\omega_n = i \frac{\pi}{2|u|} (2n+1) \quad (n \in \mathbf{Z}) \quad (\text{D.0.16})$$

originating from the term $\frac{1}{\cosh(u\omega)}$, implying the decomposition

$$\begin{aligned} \frac{1}{\cosh(u\omega)} &= \chi^+(\omega) + \chi^-(\omega), \\ \chi^+(\omega) &= \frac{i}{|u|} \sum_{n=0}^{\infty} (-1)^n \frac{1}{\omega + \omega_n}, \\ \chi^-(\omega) &= \frac{1}{\cosh(u\omega)} - \frac{i}{|u|} \sum_{n=0}^{\infty} (-1)^n \frac{1}{\omega + \omega_n}. \end{aligned} \quad (\text{D.0.17})$$

Here $\chi^+(\omega)$ and $\chi^-(\omega)$ are analytic functions in the upper and lower half-planes, respectively. With the help of eq. (D.0.17), as for any analytic and bounded function $f^-(\omega)$ in the lower half-plane, the decomposition of $\frac{f^-(\omega)}{\cosh(u\omega)}$ is

$$\begin{aligned} \frac{f^-(\omega)}{\cosh(u\omega)} &= F^+(\omega) + F^-(\omega), \\ F^+(\omega) &= \frac{i}{|u|} \sum_{n=0}^{\infty} (-1)^n \frac{f^-(-\omega_n)}{\omega + \omega_n}, \end{aligned}$$

$$F^-(\omega) = \frac{f^-(\omega)}{\cosh(u\omega)} - F^+(\omega). \quad (\text{D.0.18})$$

By virtue of eqs. (D.0.14) and (D.0.18), we make the following decomposition for $G^-(\omega)\tilde{g}_0(\omega)$,

$$\begin{aligned} Q_0^+(\omega) &= -\frac{i\mu G^-(0)}{\omega + i\varepsilon} - q(\omega) \\ Q_0^-(\omega) &= \frac{i\mu G^-(0)}{\omega + i\varepsilon} - \frac{2\pi J_1(\omega) \exp(-i\omega A) G^-(\omega)}{\omega \cosh(u\omega)} + q(\omega), \end{aligned} \quad (\text{D.0.19})$$

where $q(\omega) = 4i \sum_{n=1}^{\infty} (-1)^n \frac{G^-(-ih_n) I_1(h_n) \exp(-h_n A)}{(2n+1)(\omega + ih_n)}$, $I_1(z)$ is the first order modified Bessel function, $h_n = \frac{\pi}{2|u|}(2n+1)$, with the series converging only if $A > 1$.

If $A \gg 1$, using eq. (D.0.11), we have

$$y_0^+(\omega) = G^+(\omega) \left[-\frac{i\mu G^-(0)}{\omega + i\varepsilon} - q(\omega) \right]. \quad (\text{D.0.20})$$

Obviously, we know $y(0) = \varepsilon_1'(A) = 0$, which implies

$$0 = y(0) = \lim_{\omega \rightarrow \infty} -i\omega \tilde{y}^+(\omega). \quad (\text{D.0.21})$$

Hereafter we replace $y(\omega)$ with $y_0(\omega)$, which is a reasonable approximation if $A \gg 1$. Therefore eqs. (D.0.20) and (D.0.21) give rise to

$$\mu = -4 \sum_{n=0}^{\infty} \frac{G^-(-ih_n) I_1(h_n) \exp(-h_n A)}{(2n+1) G^-(0)}. \quad (\text{D.0.22})$$

Since we have obtained a parametric expression for the critical chemical potential, we turn to the expression for the magnetic field. Due to the fact that the phase boundary is determined by $\kappa(0) = 0$, we thus use eq. (D.0.1) to determine the magnetic field.

For simplicity, we rewrite eqs. (D.0.1) and (D.0.2) as

$$\begin{aligned} \kappa(k) &= -2 \cos k - \mu - 2u - B + \int_A^{\infty} d\Lambda [a_1(\sin k - \Lambda) + a_1(\sin k + \Lambda)] \varepsilon_1'(\Lambda) \\ &\quad - \int_{-\infty}^{\infty} d\Lambda a_1(\sin k - \Lambda) \varepsilon_1'(\Lambda), \end{aligned} \quad (\text{D.0.23})$$

$$\varepsilon_1'(\Lambda) = \varepsilon_1'^{(0)}(\Lambda) - \int_{-A}^A d\Lambda' a_2(\Lambda - \Lambda') \varepsilon_1'(\Lambda'), \quad (\text{D.0.24})$$

where we have denoted

$$\epsilon_1'^{(0)}(\Lambda) = -2\mu - 2 \int_{-\pi}^{\pi} dk \cos^2 k a_1(\sin k - \Lambda). \quad (\text{D.0.25})$$

Substituting eq. (D.0.24) into the last term on the rhs of eq. (D.0.23) gives

$$\begin{aligned} \kappa(k) = & -2 \cos k - \mu - 2u - B + \int_0^\infty d\Lambda [s(\Lambda + A - \sin k) + s(\Lambda + A + \sin k)] y(\Lambda) \\ & - \int_{-\infty}^\infty d\Lambda s(\Lambda - \sin k) \epsilon_1'^{(0)}(\Lambda), \end{aligned} \quad (\text{D.0.26})$$

where we have introduced the function $s(x) = \frac{1}{4|u| \cosh(\frac{\pi x}{2|u|})}$ and made use of the two identities eqs. (B.3.10) and (B.3.11).

Substituting the expansion $s(x) = \frac{1}{2|u|} \sum_{n=0}^\infty (-1)^n \exp(-h_n x)$, where $|\pi x/u| < 1$ and eq. (D.0.25) into eq. (D.0.26), and after some algebraic manipulations, we arrive at the result

$$\begin{aligned} \kappa(k) = & -2 \cos k - 2u - B + \sum_{n=0}^\infty \frac{(-1)^n}{|u|} \tilde{y}^+(i h_n) \cosh(h_n \sin k) \exp(-h_n A) \\ & + 2 \int_0^\infty \frac{d\omega J_1(\omega) \cos(\omega \sin k) \exp(-|u|\omega)}{\omega \cosh(u\omega)}, \end{aligned} \quad (\text{D.0.27})$$

Using eq. (D.0.27) and $\kappa(0) = 0$ we derive the expression

$$B = -2 + 2|u| + \sum_{n=0}^\infty \frac{(-1)^n}{|u|} \tilde{y}^+(i h_n) \exp(-h_n A) + 2 \int_0^\infty d\omega \frac{J_1(\omega) \exp(-|u|\omega)}{\omega \cosh(u\omega)} \quad (\text{D.0.28})$$

for determining the critical magnetic field. Here we denoted $h_n = \frac{\pi}{2|u|}(2n+1)$. The equation (D.0.28) sets up a relation between the magnetic field and the chemical potential. In summary, the phase boundary between phase IV and V is determined by eqs. (D.0.22) and (D.0.28) for $A \gg 1$.

Finite-Size Correction to the Ground State

We present the derivation of the finite-size correction to the ground state in this Appendix. Prior to this, we make clear the mathematical techniques applied, i.e., the Euler-McLaurin summation and the ‘scalar product’ relation of two sets of integral equations whose integral kernel matrices are transposed.

The Euler-McLaurin summation formula is described as

$$\frac{1}{L} \sum_{n=n_i}^{n_f} f\left(\frac{n}{L}\right) = \int_{n_-/L}^{n_+/L} dx f(x) + \frac{1}{24L^2} \left[f'\left(\frac{n_-}{L}\right) - f'\left(\frac{n_+}{L}\right) \right], \quad (\text{E.0.1})$$

where $n_- = n_i - \frac{1}{2}$, $n_+ = n_f + \frac{1}{2}$.

The detailed descriptions for the ‘scalar product’ relation states that if there are two sets of integral equations for f_i and for g_i expressed in forms of

$$\begin{aligned} f_1(x) &= f_{1,0}(x) + \int_{C_1} dx' K_{11}(x, x') f_1(x') + \int_{C_2} dy K_{12}(x, y) f_2(y), \\ f_2(y) &= f_{2,0}(y) + \int_{C_1} dx K_{21}(x, y) f_1(x) + \int_{C_2} dy' K_{22}(y, y') f_2(y'), \end{aligned}$$

and

$$\begin{aligned} g_1(x) &= g_{1,0}(x) + \int_{C_1} dx' K_{11}(x, x') g_1(x') + \int_{C_2} dy K_{21}(x, y) g_2(y), \\ g_2(y) &= g_{2,0}(y) + \int_{C_1} dx K_{12}(x, y) g_1(x) + \int_{C_2} dy' K_{22}(y, y') g_2(y'), \end{aligned}$$

then they comply with

$$\sum_{i=1,2} \int_{C_i} dz f_i(z) \cdot g_{i,0}(z) = \sum_{i=1,2} \int_{C_i} dz f_{i,0}(z) \cdot g_i(z),$$

where C_i represent different integral intervals, $f_{i,0}$ ($g_{i,0}$) are driving terms and

$K_{ij}(x, y)$ the integral kernels, which could be expressed in terms of a matrix and the kernel matrices for f_i ($i = 1, 2$) and for g_i ($i = 1, 2$) are transposed.

In order to apply the Euler-McLaurin formula, we choose a set of consecutive quantum numbers according to eq. (5.1.4)

$$I_j^u = I_-^u + j - \frac{1}{2}, \quad j = 1, 2, 3, \dots, N^u, \quad (\text{E.0.2})$$

$$I_j^b = I_-^b + j - \frac{1}{2}, \quad j = 1, 2, 3, \dots, N^b, \quad (\text{E.0.3})$$

with which we define $I_+^u = N^u + I_-^u$, and $I_+^b = N^b + I_-^b$.

Starting from eq. (5.1.5), one may rewrite this equation through denoting the driving terms of TBA equations (5.1.11) and (5.1.12) as $\epsilon_0^\gamma(k^\gamma)$ with $\gamma = u, b$

$$E = \sum_{j=1}^{N^\gamma} \epsilon_0^\gamma(k_j^\gamma). \quad (\text{E.0.4})$$

Going on with the calculation

$$\begin{aligned} \frac{E}{L} &= \frac{1}{L} \sum_{\alpha=u,b} \sum_{j=1}^{N^\alpha} \epsilon_0^\alpha(k_j^\alpha) = \frac{1}{L} \sum_{\alpha=u,b} \sum_{j=1}^{N^\alpha} \epsilon_0^\alpha \left(k^\alpha \left(\frac{2\pi I_j^\alpha}{L} \right) \right) \\ &= \frac{1}{L} \sum_{\alpha=u,b} \sum_{j=1}^{N^\alpha} \epsilon_0^\alpha \left(k^\alpha \left(\frac{I_j^\alpha}{L/(2\pi)} \right) \right) \\ &= \frac{1}{2\pi} \sum_{\alpha=u,b} \left\{ \int_{\frac{2\pi I_-^\alpha}{L}}^{\frac{2\pi I_+^\alpha}{L}} dy^\alpha \epsilon_0^\alpha(k^\alpha(y^\alpha)) - \frac{1}{24(L/2\pi)^2} \frac{d\epsilon_0^\alpha}{dy^\alpha} \Big|_{y^\alpha=\frac{2\pi I_-^\alpha}{L}}^{y^\alpha=\frac{2\pi I_+^\alpha}{L}} + \text{h.o.c.} \right\}, \end{aligned} \quad (\text{E.0.5})$$

where we have used the Euler-McLaurin formula and h.o.c. means higher order corrections. Here we use notation $k_j^\alpha = (y^\alpha)^{-1} \left(\frac{2\pi I_j^\alpha}{L} \right) = k^\alpha \left(\frac{2\pi I_j^\alpha}{L} \right)$, due to k^α is more convenient than $(y^\alpha)^{-1}$ in the change of integral variable later on. If we define $y^\alpha(Q_\pm^\alpha) = 2\pi I_\pm^\alpha/L$, then last calculation moves on to

$$\begin{aligned} \frac{E}{L} &= \frac{1}{2\pi} \sum_{\alpha=u,b} \left\{ \int_{\frac{2\pi I_-^\alpha}{L}}^{\frac{2\pi I_+^\alpha}{L}} dk^\alpha \frac{dy^\alpha}{dk^\alpha} \epsilon_0^\alpha(k^\alpha(y^\alpha)) - \frac{1}{24} \left(\frac{2\pi}{L} \right)^2 \frac{d\epsilon_0^\alpha}{dy^\alpha} \Big|_{y^\alpha=\frac{2\pi I_-^\alpha}{L}}^{y^\alpha=\frac{2\pi I_+^\alpha}{L}} + \text{h.o.c.} \right\} \\ &= \frac{1}{2\pi} \sum_{\alpha=u,b} \left\{ \int_{Q_-^\alpha}^{Q_+^\alpha} dk^\alpha 2\pi \rho_L^\alpha(k^\alpha) \epsilon_0^\alpha(k^\alpha(y^\alpha)) - \frac{1}{24} \left(\frac{2\pi}{L} \right)^2 \frac{d\epsilon_0^\alpha}{dy^\alpha} \Big|_{y^\alpha=\frac{2\pi I_-^\alpha}{L}}^{y^\alpha=\frac{2\pi I_+^\alpha}{L}} + \text{h.o.c.} \right\} \end{aligned}$$

$$= \sum_{\alpha=u,b} \left\{ \int_{Q_-^\alpha}^{Q_+^\alpha} dk^\alpha \rho_L^\alpha(k^\alpha) \epsilon_0^\alpha(k^\alpha(y^\alpha)) - \frac{1}{24} \frac{2\pi}{L^2} \frac{d\epsilon_0^\alpha}{dy^\alpha} \Big|_{y^\alpha=\frac{2\pi L^\alpha}{L}}^{y^\alpha=\frac{2\pi L_+^\alpha}{L}} + \text{h.o.c.} \right\}, \quad (\text{E.0.6})$$

where we have employed

$$\frac{dy^\alpha}{dk^\alpha} = 2\pi \rho_L^\alpha(k^\alpha),$$

which has been displayed in eqs. (5.1.9) and (5.1.10).

By now, we have obtained the integral equations for $\rho_L^\alpha(k^\alpha)$ ($\alpha = u, b$), eqs. (5.1.9) and (5.1.10), which could be expressed as a vector equation

$$\vec{\rho}_L(k^u, k^b) = \vec{\rho}_{L,0}(k^u, k^b) + \hat{K}(k^u, k^b; k, \Lambda) \odot \vec{\rho}_L(k, \Lambda), \quad (\text{E.0.7})$$

or even more simple

$$\vec{\rho}_L = \vec{\rho}_{L,0} + \hat{K} \odot \vec{\rho}_L. \quad (\text{E.0.8})$$

Here \odot means integration, and we have defined

$$\vec{\rho}_L(k^u, k^b) = \begin{bmatrix} \rho_L^u(k^u) \\ \rho_L^b(k^b) \end{bmatrix},$$

$$\vec{\rho}_{L,0}(k^u, k^b) = \begin{bmatrix} \frac{1}{2\pi} - \frac{1}{24L^2} \cos k^u \frac{a'_1(\sin k^u - k^b)}{\rho_L^b(k^b)} \Big|_{k^b=Q_-^b}^{k^b=Q_+^b} \\ \frac{1}{2\pi} \int_{-\pi}^{\pi} dk a_1(k^b - \sin k) - \frac{1}{24L^2} \left[\frac{\cos k^u a'_1(k^b - \sin(k^u))}{\rho_L^u(k^u)} \Big|_{k^u=Q_-^u}^{k^u=Q_+^u} + \frac{a'_2(k^b - k)}{\rho_L^b(k)} \Big|_{k=Q_-^b}^{k=Q_+^b} \right] \end{bmatrix},$$

and

$$\hat{K} \begin{pmatrix} k^u, k^b; k, \Lambda \end{pmatrix} = \begin{bmatrix} 0 & -\cos k^u a_1(\sin k^u - \Lambda) \\ -a_1(k^b - \sin k) & -a_2(k^b - \Lambda) \end{bmatrix}. \quad (\text{E.0.9})$$

There exist finite-size corrections in the eqs. (5.1.9) and (5.1.10) for $\rho_{L,0}^\alpha(k^\alpha)$. In general, the driving term $\rho_{L,0}^\alpha(k^\alpha)$ could be divided into a sum of result under thermodynamic limit and correction term, so does the solution to the integral equation for $\rho_L^\alpha(k^\alpha)$. Hence we write down the vector equations

$$\vec{\rho}_L(k^u, k^b) = \vec{\rho}(k^u, k^b) + \vec{f}(k^u, k^b) \quad (\text{E.0.10})$$

where $\vec{\rho}(k^u, k^b)$ and $\vec{f}(k^u, k^b)$ comply with

$$\vec{\rho}(k^u, k^b) = \vec{\rho}_0(k^u, k^b) + \hat{K}(k^u, k^b; k, \Lambda) \odot \vec{\rho}(k, \Lambda), \quad (\text{E.0.11})$$

$$\vec{f}(k^u, k^b) = \vec{f}_0(k^u, k^b) + \hat{K}(k^u, k^b; k, \Lambda) \odot \vec{f}(k^u, k^b). \quad (\text{E.0.12})$$

As like we have stated before, $\vec{\rho}_0$ and \vec{f}_0 represent the thermodynamic limit part and finite-size correction part of the driving term $\vec{\rho}_{L,0}$, respectively. Similarly $\vec{\rho}$ and \vec{f} stand for the corresponding part of ρ_L . The explicit expressions of $\vec{\rho}_0$ and \vec{f}_0 are

$$\vec{\rho}_0(k^u, k^b) = \left[\frac{1}{2\pi} \int_{-\pi}^{\pi} dk a_1(k^b - \sin k) \right], \quad (\text{E.0.13})$$

$$\vec{f}_0(k^u, k^b) = \left[\begin{array}{c} -\frac{1}{24L^2} \cos k^u \frac{a'_1(\sin k^u - k^b)}{\rho_L^b(k^b)} \Big|_{k^b=Q_+^b}^{k^b=Q_-^b} \\ -\frac{1}{24L^2} \left[\frac{\cos k^u a'_1(k^b - \sin(k^u))}{\rho_L^u(k^u)} \Big|_{k^u=Q_-^u}^{k^u=Q_+^u} + \frac{a'_2(k^b - k)}{\rho_L^b(k)} \Big|_{k=Q_-^b}^{k=Q_+^b} \right] \end{array} \right]. \quad (\text{E.0.14})$$

The TBA equations in the ground state eqs. (5.1.11) and (5.1.12) can be rewritten as a vector equation

$$\vec{\epsilon}(k^u, k^b) = \vec{\epsilon}_0(k^u, k^b) + \hat{K}^T(k^u, k^b; k, \Lambda) \odot \vec{\epsilon}(k, \Lambda) \quad (\text{E.0.15})$$

or

$$\vec{\epsilon} = \vec{\epsilon}_0 + \hat{K}^T \odot \vec{\epsilon}, \quad (\text{E.0.16})$$

where we have defined

$$\vec{\epsilon}(k^u, k^b) = \left[\begin{array}{c} \epsilon^u(k^u) \\ \epsilon^b(k^b) \end{array} \right], \quad (\text{E.0.17})$$

$$\vec{\epsilon}_0(k^u, k^b) = \left[\begin{array}{c} -2 \cos k^u - \mu - 2u - B \\ -4 \text{Re} \left[\sqrt{1 - (k^b + i \cdot |u|)^2} \right] - (2\mu + 4u) \end{array} \right], \quad (\text{E.0.18})$$

and

$$\hat{K}^T(k^u, k^b; k, \Lambda) = \left[\begin{array}{cc} 0 & -a_1(\sin k^u - \Lambda) \\ -\cos k a_1(k^b - \sin k) & -a_2(k^b - \Lambda) \end{array} \right]. \quad (\text{E.0.19})$$

Here $\hat{K}^T(k^u, k^b; k, \Lambda)$ is the transpose of $\hat{K}(k^u, k^b; k, \Lambda)$.

It's easy to notice that applying the 'scalar product' relation to eqs. (E.0.11), (E.0.12) and (E.0.15) will simplify eq. (E.0.6),

$$\begin{aligned} \frac{E}{L} &= \sum_{\alpha=u,b} \left\{ \int_{Q_-^\alpha}^{Q_+^\alpha} dk^\alpha \epsilon_0^\alpha(k^\alpha) \rho_L^\alpha(k^\alpha) - \frac{1}{24} \frac{2\pi}{L^2} \left[\frac{d\epsilon_0^\alpha}{dk^\alpha} \frac{dk^\alpha}{dy^\alpha} \right] \Big|_{y^\alpha=\frac{2\pi\alpha}{L}}^{y^\alpha=\frac{2\pi\alpha}{L}+} + \text{h.o.c.} \right\} \\ &= \sum_{\alpha=u,b} \left\{ \int_{Q_-^\alpha}^{Q_+^\alpha} dk^\alpha \epsilon_0^\alpha(k^\alpha) [\rho^\alpha(k^\alpha) + f^\alpha(k^\alpha)] - \frac{1}{24} \frac{2\pi}{L^2} \left[\frac{d\epsilon_0^\alpha}{dk^\alpha} / \frac{dy^\alpha}{dk^\alpha} \right] \Big|_{k^\alpha=Q_-^\alpha}^{k^\alpha=Q_+^\alpha} + \text{h.o.c.} \right\} \end{aligned}$$

$$\begin{aligned}
&= \sum_{\alpha=u,b} \left\{ \int_{Q_-^\alpha}^{Q_+^\alpha} dk^\alpha \varepsilon^\alpha(k^\alpha) [\rho_0^\alpha(k^\alpha) + f_0^\alpha(k^\alpha)] - \frac{1}{24L^2} \frac{d\varepsilon_0^\alpha}{dk^\alpha} \bigg|_{k^\alpha=Q_-^\alpha}^{k^\alpha=Q_+^\alpha} + \text{h.o.c.} \right\} \\
&= \sum_{\alpha=u,b} \left\{ \int_{Q_-^\alpha}^{Q_+^\alpha} dk^\alpha \varepsilon^\alpha(k^\alpha) \rho_0^\alpha(k^\alpha) - \frac{1}{24L^2} \frac{d\varepsilon_0^\alpha}{dk^\alpha} \bigg|_{k^\alpha=Q_-^\alpha}^{k^\alpha=Q_+^\alpha} \right\} \\
&\quad - \int_{Q_-^u}^{Q_+^u} dk^u \varepsilon^u(k^u) \frac{1}{24L^2} \cos k^u \frac{a'_1(\sin k^u - k^b)}{\rho_L^b(k^b)} \bigg|_{k^b=Q_-^b}^{k^b=Q_+^b} \\
&\quad - \int_{Q_-^b}^{Q_+^b} dk^b \varepsilon^b(k^b) \frac{1}{24L^2} \left[\frac{\cos k^u a'_1(k^b - \sin(k^u))}{\rho_L^u(k^u)} \bigg|_{k^u=Q_-^u}^{k^u=Q_+^u} + \frac{a'_2(k^b - k)}{\rho_L^b(k)} \bigg|_{k=Q_-^b}^{k=Q_+^b} \right] + \text{h.o.c.} \\
&= \sum_{\alpha=u,b} \left\{ \int_{Q_-^\alpha}^{Q_+^\alpha} dk^\alpha \varepsilon^\alpha(k^\alpha) \rho_0^\alpha(k^\alpha) - \frac{1}{24L^2} \frac{d\varepsilon_0^\alpha}{dk^\alpha} \bigg|_{k^\alpha=Q_-^\alpha}^{k^\alpha=Q_+^\alpha} \right\} \\
&\quad - \frac{1}{24L^2} \left[\frac{1}{\rho_L^b(k^b)} \int_{Q_-^u}^{Q_+^u} dk^u \varepsilon^u(k^u) \cos k^u a'_1(\sin k^u - k^b) \right] \bigg|_{k^b=Q_-^b}^{k^b=Q_+^b} \\
&\quad - \frac{1}{24L^2} \left\{ \left[\frac{\cos k^u}{\rho_L^u(k^u)} \int_{Q_-^b}^{Q_+^b} dk^b \varepsilon^b(k^b) a'_1(k^b - \sin(k^u)) \right] \bigg|_{k^u=Q_-^u}^{k^u=Q_+^u} \right. \\
&\quad \left. + \left[\frac{1}{\rho_L^b(k)} \int_{Q_-^b}^{Q_+^b} dk^b \varepsilon^b(k^b) a'_2(k^b - k) \right] \bigg|_{k=Q_-^b}^{k=Q_+^b} \right\} + \text{h.o.c.} \\
&= \sum_{\alpha=u,b} \left\{ \int_{Q_-^\alpha}^{Q_+^\alpha} dk^\alpha \varepsilon^\alpha(k^\alpha) \rho_0^\alpha(k^\alpha) - \frac{1}{24L^2} \frac{d\varepsilon_0^\alpha}{dk^\alpha} \bigg|_{k^\alpha=Q_-^\alpha}^{k^\alpha=Q_+^\alpha} \right\} \\
&\quad - \frac{1}{24L^2} \left[\frac{1}{\rho_L^b(k^b)} \int_{Q_-^u}^{Q_+^u} \left(da_1(\sin k^u - k^b) \right) \varepsilon^u(k^u) \right] \bigg|_{k^b=Q_-^b}^{k^b=Q_+^b} \\
&\quad - \frac{1}{24L^2} \left\{ \left[\frac{\cos k^u}{\rho_L^u(k^u)} \int_{Q_-^b}^{Q_+^b} \left(da_1(k^b - \sin(k^u)) \right) \varepsilon^b(k^b) \right] \bigg|_{k^u=Q_-^u}^{k^u=Q_+^u} \right. \\
&\quad \left. + \left[\frac{1}{\rho_L^b(k)} \int_{Q_-^b}^{Q_+^b} \left(da_2(k^b - k) \right) \varepsilon^b(k^b) \right] \bigg|_{k=Q_-^b}^{k=Q_+^b} \right\} + \text{h.o.c.} \\
&= \sum_{\alpha=u,b} \left\{ \int_{Q_-^\alpha}^{Q_+^\alpha} dk^\alpha \varepsilon^\alpha(k^\alpha) \rho_0^\alpha(k^\alpha) - \frac{1}{24L^2} \frac{d\varepsilon_0^\alpha}{dk^\alpha} \bigg|_{k^\alpha=Q_-^\alpha}^{k^\alpha=Q_+^\alpha} \right\}
\end{aligned}$$

$$\begin{aligned}
& -\frac{1}{24L^2} \left[\frac{1}{\rho_L^b(k^b)} \left(a_1(\sin k^u - k^b) \epsilon^u(k^u) \right) \Big|_{k^u=Q_-^u}^{k^u=Q_+^u} - \frac{1}{\rho_L^b(k^b)} \int_{Q_-^u}^{Q_+^u} dk^u a_1(\sin k^u - k^b) \frac{d\epsilon^u(k^u)}{dk^u} \right] \Big|_{k^b=Q_-^b}^{k^b=Q_+^b} \\
& -\frac{1}{24L^2} \left\{ \left[\frac{\cos k^u}{\rho_L^u(k^u)} \left(a_1(k^b - \sin(k^u)) \epsilon^b(k^b) \right) \Big|_{k^b=Q_-^b}^{k^b=Q_+^b} - \frac{\cos k^u}{\rho_L^u(k^u)} \int_{Q_-^b}^{Q_+^b} dk^b a_1(k^b - \sin(k^u)) \frac{d\epsilon^b(k^b)}{dk^b} \right] \Big|_{k^u=Q_-^u}^{k^u=Q_+^u} \right. \\
& \left. + \left[\frac{1}{\rho_L^b(k)} \left(a_2(k^b - k) \epsilon^b(k^b) \right) \Big|_{k^b=Q_-^b}^{k^b=Q_+^b} - \frac{1}{\rho_L^b(k)} \int_{Q_-^b}^{Q_+^b} dk^b a_2(k^b - k) \frac{d\epsilon^b(k^b)}{dk^b} \right] \Big|_{k=Q_-^b}^{k=Q_+^b} \right\} + \text{h.o.c.}, \tag{E.0.20}
\end{aligned}$$

where using $Q_+^\alpha = -Q_-^\alpha$, $\epsilon^\alpha(-k^\alpha) = \epsilon^\alpha(k^\alpha)$ and $\epsilon^\alpha(Q_\pm^\alpha) = 0$ in the ground state one can further obtain that

$$\begin{aligned}
\frac{E}{L} &= \sum_{\alpha=u,b} \left\{ \int_{Q_-^\alpha}^{Q_+^\alpha} dk^\alpha \epsilon^\alpha(k^\alpha) \rho_0^\alpha(k^\alpha) \right\} \\
& -\frac{1}{24L^2} \left\{ \frac{1}{\rho_L^u(k^u)} \left[\frac{d\epsilon_0^u}{dk^u} - \cos k^u \int_{Q_-^b}^{Q_+^b} dk^b a_1(k^b - \sin k^u) \frac{d\epsilon^b(k^b)}{dk^b} \right] \right\} \Big|_{k^u=Q_-^u}^{k^u=Q_+^u} \\
& -\frac{1}{24L^2} \left\{ \frac{1}{\rho_L^b(k^b)} \left[\frac{d\epsilon_0^b}{dk^b} - \int_{Q_-^u}^{Q_+^u} dk^u a_1(\sin k^u - k^b) \frac{d\epsilon^u(k^u)}{dk^u} - \int_{Q_-^b}^{Q_+^b} dk a_2(k^b - k) \frac{d\epsilon^b(k)}{dk} \right] \right\} \Big|_{k^b=Q_-^b}^{k^b=Q_+^b} \\
&= \sum_{\alpha=u,b} \left\{ \int_{Q_-^\alpha}^{Q_+^\alpha} dk^\alpha \epsilon^\alpha(k^\alpha) \rho_0^\alpha(k^\alpha) - \frac{1}{24L^2} \frac{\frac{d\epsilon^\alpha}{dk^\alpha}}{\rho_L^\alpha(k^\alpha)} \Big|_{k^\alpha=Q_-^\alpha}^{k^\alpha=Q_+^\alpha} \right\} + \text{h.o.c.} \\
&= \sum_{\alpha=u,b} \left\{ \int_{Q_-^\alpha}^{Q_+^\alpha} dk^\alpha \epsilon^\alpha(k^\alpha) \rho_0^\alpha(k^\alpha) - \frac{1}{24L^2} \frac{\frac{d\epsilon^\alpha}{dk^\alpha}}{\rho^\alpha(k^\alpha)} \Big|_{k^\alpha=Q_-^\alpha}^{k^\alpha=Q_+^\alpha} \right\} + \text{h.o.c.} \tag{E.0.21}
\end{aligned}$$

Under the thermodynamic limit one knows that the dressed momentum $p^\gamma(k^\gamma) = \lim_{L \rightarrow \infty} y_L^\gamma(k^\gamma)$ with $\gamma = u, b$. In the ground state, the sound velocity is defined by $v^\gamma = \pm \frac{d\epsilon^\gamma(k^\gamma)}{dp^\gamma(k^\gamma)} \Big|_{k^\gamma=\pm Q_\gamma} = \pm \frac{d\epsilon^\gamma/dk^\gamma}{2\pi\rho^\gamma(k^\gamma)} \Big|_{k^\gamma=\pm Q_\gamma}$. Thereby one can deduce that

$$\frac{E}{L} = \sum_{\alpha=u,b} \left\{ \int_{Q_-^\alpha}^{Q_+^\alpha} dk^\alpha \epsilon^\alpha(k^\alpha) \rho_0^\alpha(k^\alpha) - \frac{\pi}{6L^2} v^\alpha \right\} + \text{h.o.c.} \tag{E.0.22}$$

where the integrals on the rhs are the ground state energy in the thermodynamic limit and can be easily derived through eq. (E.0.4).

Obviously, the finite-size correction to the ground state energy is given by

$$\Delta\left(\frac{E}{L}\right) = - \sum_{\alpha=u,b} \frac{\pi}{6L^2} v^\alpha. \quad (\text{E.0.23})$$

In the ground state, the momentum distribution is symmetric whether under the thermodynamic limit or not, therefore the finite-size correction to the total momentum is zero, i.e.

$$\Delta P = 0. \quad (\text{E.0.24})$$

Finite-Size Correction to Three Types of Elementary Excitations

The situation of Type I elementary excitation is simple, which is a special case of particle-hole excitation. The change in energy of a particle-hole excitation is expressed by

$$\Delta E = \sum_{\alpha=u,b} \sum_{\beta=+,-} \left[\epsilon^\alpha(k_{p,\beta}^\alpha) - \epsilon^\alpha(k_{h,\beta}^\alpha) \right], \quad (\text{F.0.1})$$

where we use subscripts p and h to label particle and hole respectively. Due to the Type I excitation taking place close to the Fermi points, one can approximate the difference in the last equation as the leading term of Taylor expansion around the Fermi points. Hence, the change in energy of Type I excitation is written as

$$\Delta E \approx \sum_{\alpha=u,b} \sum_{\beta=+,-} \left[\left. \frac{\partial \epsilon^\alpha}{\partial k^\alpha} \right|_{k^\alpha=Q_\beta^\alpha} \left(k_{p,\beta}^\alpha - k_{h,\beta}^\alpha \right) \right]. \quad (\text{F.0.2})$$

Moreover, noticing that the counting function connects the momentum k^α and quantum number I^α , we similarly introduce Taylor expansion in the last equation, and obtain

$$\begin{aligned} \Delta E &\approx \sum_{\alpha=u,b} \sum_{\beta=+,-} \left[\left. \frac{\partial \epsilon^\alpha}{\partial k^\alpha} \right|_{k^\alpha=Q_\beta^\alpha} \left. \frac{dk^\alpha}{dy^\alpha} \right|_{k^\alpha=Q_\beta^\alpha} \frac{2\pi}{L} N_\beta^\alpha \text{sign}(\beta) \right] \\ &= \sum_{\alpha=u,b} \frac{2\pi}{L} v^\alpha (N_+^\alpha + N_-^\alpha), \end{aligned} \quad (\text{F.0.3})$$

where y^α is the counting function under the thermodynamic limit, i.e., $y^\alpha = \lim_{L \rightarrow \infty} y_L^\alpha(k^\alpha) = p^\alpha(k^\alpha)$, and we have used the definition of sound velocity $v^\gamma = \pm \left. \frac{d\epsilon^\gamma(k^\gamma)/dk^\gamma}{dp^\gamma(k^\gamma)/dk^\gamma} \right|_{k^\gamma=\pm Q^\gamma}$.

In light of the zero momentum in the ground state, we can write down the change in total momentum

$$\Delta P = \sum_{\alpha=u,b} \frac{2\pi}{L} (N_+^\alpha - N_-^\alpha), \quad (\text{F.0.4})$$

where $N_+^\alpha = I_{+,e}^\alpha - I_{-,g}^\alpha$ and $N_-^\alpha = I_{-,g}^\alpha - I_{+,e}^\alpha$ arise from the change in distribution of quantum numbers close to the right and left Fermi points, respectively. The subscripts e and g represent the excited state and ground state, respectively.

On the other hand, the derivations for the changes in energy and total momentum of Type II and III elementary excitations are rather complicated. Hereby we give a sketch of calculations in this Appendix. The low-lying excitation corresponds to small variations in Fermi points from the ground state case. This allows us to expand the energy per site with respect to the changes of the Fermi points

$$\begin{aligned} & e_E(Q_\pm^u, Q_\pm^b) \\ &= e_E(\pm Q^u, \pm Q^b) + \frac{1}{2} \sum_{\alpha=u,b} \left[\frac{\partial^2 e_E}{\partial (Q_+^\alpha)^2} \Big|_G (Q_+^\alpha - Q^\alpha)^2 + \frac{\partial^2 e_E}{\partial (Q_-^\alpha)^2} \Big|_G (Q_-^\alpha + Q^\alpha)^2 \right] \\ &+ \text{h.o.c.} \end{aligned} \quad (\text{F.0.5})$$

where $e_E = \sum_{\alpha=u,b} \int_{Q_-^\alpha}^{Q_+^\alpha} dk^\alpha \epsilon^\alpha(k^\alpha) \rho_0^\alpha(k^\alpha)$ is the energy per site, $\pm Q_\alpha$ stand for the ground state Fermi points, and subscript G means $Q_\pm^u = \pm Q_u$ & $Q_\pm^b = \pm Q_b$. In this expansion, the first order derivative $\frac{\partial e_E}{\partial (Q_\pm^\alpha)} \Big|_G$ vanishes due to the fact that the Fermi points minimize the energy in the ground state.

With the help of TBA equations and integral equations of root densities, the second order derivative is given by

$$\frac{\partial^2 e_E}{\partial (Q_\pm^\alpha)^2} \Big|_G = \pm \frac{\partial \epsilon^\alpha(k^\alpha)}{\partial Q_\pm^\alpha} \Big|_{G, k^\alpha = \pm Q_\alpha} \rho^\alpha(\pm Q_\alpha), \quad (\text{F.0.6})$$

substituting which and sound velocity $v^\gamma = \pm \frac{d\epsilon^\gamma(k^\gamma)/dk^\gamma}{2\pi\rho^\gamma(k^\gamma)} \Big|_G$ into eq. (F.0.5) yields

$$\begin{aligned} & e_E(Q_\pm^u, Q_\pm^b) \\ &= e_E(\pm Q^u, \pm Q^b) + \pi \sum_{\alpha=u,b} v^\alpha [\rho^\alpha(Q^\alpha)]^2 [(Q_+^\alpha - Q^\alpha)^2 + (Q_-^\alpha + Q^\alpha)^2] + \text{h.o.c.} \end{aligned} \quad (\text{F.0.7})$$

In order to rewrite $Q_+^\gamma - Q^\gamma$ and $Q_-^\gamma + Q^\gamma$ in terms of ΔN^γ and ΔD^γ , we introduce notations $\mathbf{v}^\gamma = \frac{I_+^\gamma - I_-^\gamma}{L} = \frac{N^\gamma}{L}$ and $\delta^\gamma = \frac{I_+^\gamma + I_-^\gamma}{2L} = \frac{D^\gamma}{L}$, where N^u (N^b) stands for the

number of unpaired fermions (bound pairs), and D^γ is the position of the center of the Fermi sea for unpaired fermions (bound pairs). Their explicit expression are respectively given by

$$\mathbf{v}^\gamma = \int_{Q_-^\gamma}^{Q_+^\gamma} dk \rho^\gamma(k) \quad (\gamma = u, b) \quad (\text{F.0.8})$$

$$\delta^u = \frac{1}{2} \left(\int_{\pi}^{Q_+^u} + \int_{-\pi}^{Q_-^u} \right) dk \rho^u(k) + \frac{1}{2\pi} \int_{Q_-^b}^{Q_+^b} dk^b \theta \left(\frac{k^b}{|u|} \right) \rho^b(k^b), \quad (\text{F.0.9})$$

$$\delta^b = \frac{1}{2} \left(\int_{\infty}^{Q_+^b} + \int_{-\infty}^{Q_-^b} \right) dk \rho^b(k), \quad (\text{F.0.10})$$

with $\theta(x) = 2 \arctan(x)$.

The total differential of \mathbf{v}^γ ($\gamma = u, b$) with respect to Q_\pm^β ($\beta = u, b$) in the vicinity of ground state is expressed by

$$d\mathbf{v}^\gamma = \sum_{\beta=u,b} \left[\left. \frac{\partial \mathbf{v}^\gamma}{\partial Q_+^\beta} \right|_G dQ_+^\beta + \left. \frac{\partial \mathbf{v}^\gamma}{\partial Q_-^\beta} \right|_G dQ_-^\beta \right]. \quad (\text{F.0.11})$$

We furthermore denote

$$d\vec{\mathbf{v}} = \begin{bmatrix} d\mathbf{v}^u \\ d\mathbf{v}^b \end{bmatrix}, \quad d\vec{Q}_\pm = \begin{bmatrix} dQ_\pm^u \\ dQ_\pm^b \end{bmatrix}, \quad \{\hat{V}_\pm\}_{\mathcal{V}\beta} = \left. \frac{\partial \mathbf{v}^\gamma}{\partial Q_\pm^\beta} \right|_G, \quad (\text{F.0.12})$$

and then eq. (F.0.11) could be rewritten as a vector equation

$$d\vec{\mathbf{v}} = \hat{V}_+ d\vec{Q}_+ + \hat{V}_- d\vec{Q}_-. \quad (\text{F.0.13})$$

Here we need to calculate

$$\frac{\partial \mathbf{v}^\gamma}{\partial Q_\pm^\beta} = \int_{Q_-^\gamma}^{Q_+^\gamma} dk \frac{\partial \rho^\gamma(k)}{\partial Q_\pm^\beta} \pm \rho^\gamma(Q_\pm^\beta) \delta_{\gamma\beta}. \quad (\text{F.0.14})$$

According to eq. (E.0.11), it is easy to derive that

$$\frac{\partial \rho^\gamma(k^\gamma)}{\partial Q_\pm^\beta} = \pm \hat{K}_{\gamma\beta}(k^\gamma; k^\beta = Q_\pm^\beta) \rho^\beta(Q_\pm^\beta) + \sum_{\eta=u,b} \hat{K}_{\gamma\eta}(k^\gamma; k^\eta) \odot \frac{\partial \rho^\eta(k^\eta)}{\partial Q_\pm^\beta}, \quad (\text{F.0.15})$$

which together with the dressed charge equations brings us that

$$\frac{\partial \mathbf{v}^\gamma}{\partial Q_\pm^\beta} = \pm \rho^\beta(Q_\pm^\beta) Z_{\gamma\beta}(k^\beta = Q_\pm^\beta). \quad (\text{F.0.16})$$

We introduce the following notations

$$\hat{Z} = \left[\begin{array}{cc} Z_{uu}(k^u = Q_+^u) & Z_{ub}(k^b = Q_+^b) \\ Z_{bu}(k^u = Q_+^u) & Z_{bb}(k^b = Q_+^b) \end{array} \right] \Big|_G, \quad (\text{F.0.17})$$

$$\hat{\mathbf{p}} = \left[\begin{array}{cc} \mathbf{p}^u(k^u = Q_+^u) & 0 \\ 0 & \mathbf{p}^b(k^b = Q_+^b) \end{array} \right] \Big|_G, \quad (\text{F.0.18})$$

and inserting which into eq. (F.0.16), then we have a compact expression

$$\hat{V}_\pm = \pm \hat{Z} \cdot \hat{\mathbf{p}}. \quad (\text{F.0.19})$$

Now we can express eq. (F.0.13) as

$$d\vec{V} = \hat{Z} \cdot \vec{\mathbf{p}} \cdot d\vec{Q}_+ - \hat{Z} \cdot \vec{\mathbf{p}} \cdot d\vec{Q}_-. \quad (\text{F.0.20})$$

In the next stage, we similarly consider the total differential of δ^γ with respect to Q_\pm^β in the vicinity of the ground state

$$d\delta^\gamma = \sum_{\beta=u,b} \left[\frac{\partial \delta^\gamma}{\partial Q_+^\beta} \Big|_G dQ_+^\beta + \frac{\partial \delta^\gamma}{\partial Q_-^\beta} \Big|_G dQ_-^\beta \right]. \quad (\text{F.0.21})$$

Similarly, we introduce notations

$$d\vec{\delta} = \left[\begin{array}{c} d\delta^u \\ d\delta^b \end{array} \right], \quad \{\hat{W}_\pm\}_{\gamma\beta} = \frac{\partial \delta^\gamma}{\partial Q_\pm^\beta} \Big|_G, \quad (\text{F.0.22})$$

and then eq. (F.0.21) can be rewritten as a vector equation

$$d\vec{\delta} = \hat{W}_+ d\vec{Q}_+ + \hat{W}_- d\vec{Q}_-. \quad (\text{F.0.23})$$

We further calculate

$$\begin{aligned} \frac{\partial \delta^u}{\partial Q_\pm^\beta} &= \frac{1}{2} \mathbf{p}^u(Q_\pm^u) \cdot \delta_{u\beta} \pm \frac{1}{2} \theta \left(\frac{Q_\pm^b}{|u|} \right) \mathbf{p}^b(Q_\pm^b) \cdot \delta_{b\beta} \\ &\quad + \frac{1}{2} \left(\int_\pi^{Q_+^u} + \int_{-\pi}^{Q_-^u} \right) dk^u \frac{\partial \mathbf{p}^u(k^u)}{\partial Q_\pm^\beta} + \frac{1}{2\pi} \int_{Q_-^b}^{Q_+^b} dk^b \theta \left(\frac{k^b}{|u|} \right) \frac{\partial \mathbf{p}^b(k^b)}{\partial Q_\pm^\beta} \end{aligned} \quad (\text{F.0.24})$$

$$\frac{\partial \delta^b}{\partial Q_\pm^\beta} = \frac{1}{2} \mathbf{p}^b(Q_\pm^b) \cdot \delta_{b\beta} + \frac{1}{2} \left(\int_\infty^{Q_+^b} + \int_{-\infty}^{Q_-^b} \right) dk^b \frac{\partial \mathbf{p}^b(k^b)}{\partial Q_\pm^\beta}. \quad (\text{F.0.25})$$

Substituting eq. (F.0.15) into eqs. (F.0.24) and (F.0.25), we obtain the fol-

lowing expressions

$$\begin{aligned} \frac{\partial \delta^u}{\partial Q_{\pm}^{\beta}} = & \pm \frac{1}{2} \rho^{\beta}(Q_{\pm}^{\beta}) \left[\left(\int_{\pi}^{Q_{+}^u} + \int_{-\pi}^{Q_{-}^u} \right) dk^u \hat{K}_{u\beta}(k^u; k^{\beta} = Q_{\pm}^{\beta}) \pm \delta_{u\beta} + \frac{1}{\pi} \theta \left(\frac{Q_{\pm}^b}{|u|} \right) \cdot \delta_{b\beta} \right] \\ & + \frac{1}{2\pi} \int_{Q_{-}^b}^{Q_{+}^b} dk^b \frac{\partial \rho^b(k^b)}{\partial Q_{\pm}^{\beta}} \left[\theta \left(\frac{k^b}{|u|} \right) + \pi \left(\int_{\pi}^{Q_{+}^u} + \int_{-\pi}^{Q_{-}^u} \right) dk^u \hat{K}_{ub}(k^u; k^b) \right], \end{aligned} \quad (\text{F.0.26})$$

$$\begin{aligned} \frac{\partial \delta^b}{\partial Q_{\pm}^{\beta}} = & \pm \frac{1}{2} \rho^b(Q_{\pm}^b) \left[\left(\int_{\infty}^{Q_{+}^b} + \int_{-\infty}^{Q_{-}^b} \right) dk^b \hat{K}_{b\beta}(k^b; k^{\beta} = Q_{\pm}^{\beta}) \pm \delta_{b\beta} \right] \\ & + \frac{1}{2} \sum_{\gamma=u,b} \int_{Q_{-}^{\gamma}}^{Q_{+}^{\gamma}} dk^{\gamma} \frac{\partial \rho^{\gamma}(k^{\gamma})}{\partial Q_{\pm}^{\beta}} \left[\left(\int_{\infty}^{Q_{+}^b} + \int_{-\infty}^{Q_{-}^b} \right) dk^b \hat{K}_{b\gamma}(k^b; k^{\gamma}) \right]. \end{aligned} \quad (\text{F.0.27})$$

Now we introduce a set of new integral equations similar to the dressed charge equations

$$\begin{aligned} \sigma_{u\eta}(k^{\eta}) = & \delta_{b\eta} \cdot \left[\frac{1}{\pi} \theta \left(\frac{k^b}{|u|} \right) + \left(\int_{\pi}^{Q_{+}^u} + \int_{-\pi}^{Q_{-}^u} \right) dk^u \hat{K}_{ub}(k^u; k^b) \right] \\ & + \sum_{\gamma=u,b} \hat{K}_{\eta\gamma}^T(k^{\eta}; k^{\gamma}) \odot \sigma_{u\gamma}(k^{\gamma}), \end{aligned} \quad (\text{F.0.28})$$

$$\sigma_{b\eta}(k^{\eta}) = \left(\int_{\infty}^{Q_{+}^b} + \int_{-\infty}^{Q_{-}^b} \right) d\tilde{k}^b \hat{K}_{b\beta}(\tilde{k}^b; k^{\eta}) + \sum_{\gamma=u,b} \hat{K}_{\eta\gamma}^T(k^{\eta}; \tilde{k}^{\gamma}) \odot \sigma_{b\gamma}(\tilde{k}^{\gamma}), \quad (\text{F.0.29})$$

where $\eta = u, b$, and see the integral kernel matrix \hat{K}^T in eq. (E.0.19).

With the help of the above integral equations for $\sigma_{\alpha\beta}(k^{\beta})$ and eq. (F.0.15), we obtain

$$\frac{\partial \delta^{\alpha}}{\partial Q_{\pm}^{\beta}} = \pm \frac{1}{2} \rho^{\beta}(Q_{\pm}^{\beta}) \left[\sigma_{\alpha\beta}(Q_{\pm}^{\beta}) \pm \delta_{\alpha\beta} \right], \quad (\text{F.0.30})$$

and

$$\{\hat{W}\}_{\pm} = \left\{ \pm \frac{1}{2} \rho^{\beta}(Q_{\pm}^{\beta}) \left[\sigma_{\alpha\beta}(Q_{\pm}^{\beta}) \pm \delta_{\alpha\beta} \right] \right\} \Big|_G. \quad (\text{F.0.31})$$

We now further define

$$\hat{\sigma} = \left[\begin{array}{cc} \sigma_{uu}(Q_{+}^u) & \sigma_{ub}(Q_{+}^b) \\ \sigma_{bu}(Q_{+}^u) & \sigma_{bb}(Q_{+}^b) \end{array} \right] \Big|_G, \quad (\text{F.0.32})$$

and hereby eq. (F.0.31) can be cast into a vector form

$$\hat{W}_{\pm} = \frac{1}{2} (\hat{\sigma} + \hat{I}) \cdot \hat{\mathbf{p}}. \quad (\text{F.0.33})$$

Moreover, $\hat{\sigma}$ could be expressed in terms of \hat{Z} . Taking the derivative of eqs. (F.0.28) and (F.0.29) with respect to their own arguments, we then obtain

$$\begin{aligned} \sigma'_{u\eta}(k^{\eta}) &= 2a_1(k^b) \delta_{b\eta} - \hat{K}_{u\eta}^T(k^u; k^{\eta}) \Big|_{k^u=\pi}^{k^u=Q_+^{\pi}} - \hat{K}_{u\eta}^T(k^u; k^{\eta}) \Big|_{k^u=-\pi}^{k^u=Q_-^{\pi}} \\ &\quad - \sum_{\gamma=u,b} \left[\hat{K}_{\eta\gamma}^T(k^{\eta}; k^{\gamma}) \sigma_{u\gamma}(k^{\gamma}) \right] \Big|_{k^{\gamma}=Q_-^{\gamma}}^{k^{\gamma}=Q_+^{\gamma}} + \sum_{\gamma=u,b} \hat{K}_{\eta\gamma}(k^{\eta}; k^{\gamma}) \odot \sigma'_{u\gamma}(k^{\gamma}), \end{aligned} \quad (\text{F.0.34})$$

$$\begin{aligned} \sigma'_{b\eta}(k^{\eta}) &= - \hat{K}_{b\eta}^T(k^{\eta}; k^b) \Big|_{k^b=\infty}^{k^b=Q_+^b} - \hat{K}_{b\eta}^T(k^{\eta}; k^b) \Big|_{k^b=-\infty}^{k^b=Q_-^b} \\ &\quad - \sum_{\gamma=u,b} \left[\hat{K}_{\eta\gamma}^T(k^{\eta}; k^{\gamma}) \sigma_{b\gamma}(k^{\gamma}) \right] \Big|_{k^{\gamma}=Q_-^{\gamma}}^{k^{\gamma}=Q_+^{\gamma}} + \sum_{\gamma=u,b} \hat{K}_{\eta\gamma}(k^{\eta}; k^{\gamma}) \odot \sigma'_{b\gamma}(k^{\gamma}). \end{aligned} \quad (\text{F.0.35})$$

Taking the integral of $\sigma'_{u\eta}(k^{\eta})$ with respect to k^{η} over interval $[Q_-^{\eta}, Q_+^{\eta}]$ leads to

$$\delta_{u\beta} - \hat{Z}_{u\beta}^T - \sum_{\gamma=u,b} \hat{\sigma}_{u\gamma} \cdot \hat{Z}_{\gamma\beta}^T = 0. \quad (\text{F.0.36})$$

Similarly, one can derive the following equation through integral of $\sigma'_{b\eta}(k^{\eta})$,

$$\delta_{b\beta} - \hat{Z}_{b\beta}^T - \sum_{\gamma=u,b} \hat{\sigma}_{b\gamma} \cdot \hat{Z}_{\gamma\beta}^T = 0. \quad (\text{F.0.37})$$

We rewrite the above two equations as a vector equation,

$$\hat{I} - \hat{Z}^T - \hat{\sigma} \cdot \hat{Z}^T = 0, \quad (\text{F.0.38})$$

inserting which into eq. (F.0.33) yields

$$\hat{W}_{\pm} = \frac{1}{2} (\hat{Z}^T)^{-1} \cdot \hat{\mathbf{p}}. \quad (\text{F.0.39})$$

Using the last equation, one can express eq. (F.0.23) as

$$d\vec{\sigma} = \frac{1}{2} (\hat{Z}^T)^{-1} \cdot \hat{\mathbf{p}} \cdot d\vec{Q}_+ - \frac{1}{2} (\hat{Z}^T)^{-1} \cdot \hat{\mathbf{p}} \cdot d\vec{Q}_-. \quad (\text{F.0.40})$$

Now we rewrite eqs. (F.0.13) and (F.0.40) as

$$\begin{bmatrix} d\vec{v} \\ d\vec{\delta} \end{bmatrix} = \begin{bmatrix} \hat{Z} & -\hat{Z} \\ \frac{1}{2}(\hat{Z}^T)^{-1} & \frac{1}{2}(\hat{Z}^T)^{-1} \end{bmatrix} \cdot \begin{bmatrix} \hat{\rho} d\vec{Q}_+ \\ \hat{\rho} d\vec{Q}_- \end{bmatrix}, \quad (\text{F.0.41})$$

whose inverse is given by

$$\begin{bmatrix} \hat{\rho} d\vec{Q}_+ \\ \hat{\rho} d\vec{Q}_- \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\hat{Z}^{-1} & \hat{Z}^T \\ -\frac{1}{2}\hat{Z}^{-1} & \hat{Z}^T \end{bmatrix} \cdot \begin{bmatrix} d\vec{v} \\ d\vec{\delta} \end{bmatrix}. \quad (\text{F.0.42})$$

At last, we express eq. (F.0.7) in terms of dQ_\pm^γ ($\gamma = u, b$)

$$\begin{aligned} & e_E(Q_\pm^u, Q_\pm^b) - e_E(\pm Q^u, \pm Q^b) \\ &= \pi \sum_{\gamma=u,b} v^\gamma [\rho^\gamma(Q^\gamma)]^2 [(Q_+^\gamma - Q^\gamma)^2 + (Q_-^\gamma + Q^\gamma)^2] + \text{h.o.c.} \\ &= \pi \sum_{\gamma=u,b} v^\gamma [\rho^\gamma(Q^\gamma)]^2 [(dQ_+^\gamma)^2 + (dQ_-^\gamma)^2] + \text{h.o.c.} \end{aligned} \quad (\text{F.0.43})$$

In light of the definitions of \hat{S} , $\hat{\rho}$, $\Delta\vec{N}$, $\Delta\vec{D}$, and $v^\gamma = \frac{N^\gamma}{L}$, $\delta^\gamma = \frac{D^\gamma}{L}$ ($\gamma = u, b$), one could recast the last equation into

$$\begin{aligned} & e_E(Q_\pm^u, Q_\pm^b) - e_E(\pm Q^u, \pm Q^b) \\ &= \frac{2\pi}{L^2} \left[\frac{1}{4} (\Delta\vec{N})^T \cdot (\hat{Z}^{-1})^T \cdot \hat{S}_v \cdot \hat{Z}^{-1} \cdot \Delta\vec{N} + (\Delta\vec{D})^T \cdot \hat{Z} \cdot \hat{S}_v \cdot \hat{Z}^T \cdot \Delta\vec{D} \right] + \text{h.o.c.} \end{aligned} \quad (\text{F.0.44})$$

The change in total momentum caused by Type II and III excitations is given by

$$\begin{aligned} \Delta P &= \sum_{\gamma=u,b} \sum_{j=1}^{N^\gamma} \frac{2\pi I_j^\gamma}{L} \\ &= \sum_{\gamma=u,b} \frac{2\pi}{L} \frac{1}{2} (I_+^\gamma - I_-^\gamma) (I_N^\gamma + I_1^\gamma) \\ &= \frac{2\pi}{L} \sum_{\gamma=u,b} \Delta D^\gamma \cdot (N^\gamma + \Delta N^\gamma). \end{aligned} \quad (\text{F.0.45})$$

After the above lengthy calculations, the changes in energy and total momentum of the three types of elementary excitations have been derived. The change in energy shown in eqs. (5.2.2) and (F.0.44) is summarized as eq. (5.2.3). Whereas the change in total momentum shown in eqs. (5.2.1) and (F.0.45) is summarized as eq. (5.2.4).

Bibliography

- [1] J. Hubbard, J. Proc. R. Soc. A 276,238 (1963).
- [2] J. Hubbard, J. Proc. R. Soc. A 281 401 (1964).
- [3] M. C. Gutzwiller, Phys. Rev. Lett. 10, 159 (1963).
- [4] J. Kanamori, Prog. Theo. Phys. 30 275 (1963).
- [5] M. Rasetti, The Hubbard Model Recent Results, (World Scientific, 1991).
- [6] A Montorsi, The Hubbard Model: A Collection Of Reprints, (World Scientific, 1992).
- [7] D. Baeriswyl, D. K. Campbell, J. M. P. Carmelo, F. Guinea, E. Louis, The Hubbard Model-Its Physics and Mathematical Physics, (Springer, 1995).
- [8] F. H. L. Essler, H. Frahm, F. Göhmann, A. Klümper and V. E. Korepin, The One-Dimensional Hubbard Model (Cambridge University Press, Cambridge, 2005).
- [9] C. N. Yang, Phys. Rev. Lett. 19, 1312 (1967).
- [10] R. J. Baxter, Ann. Phys. (N.Y.) 70, 193 (1972).
- [11] E. H. Lieb and F. Y. Wu, Phys. Rev. Lett. 20, 1445 (1968).
- [12] B. S. Shastry, Phys. Rev. Lett. 56, 2453 (1986).
- [13] M. Takahashi, Prog. Theo. Phys. 42, 1098 (1969).
- [14] M. Takahashi, Prog. Theo. Phys. 43, 860 (1970).
- [15] M. Takahashi, Prog. Theo. Phys. 43, 1619 (1970).
- [16] M. Takahashi, Prog. Theo. Phys. 45, 756 (1971).
- [17] H. Shiba, Phys. Rev. B 6, 930 (1972).
- [18] V. Ya. Krivnov and A. A. Ovchinnikov, Zh. Eksp. Teor. Fiz. 67, 1568 (1974).
- [19] T. Usuki, N. Kawakami and A. Okiji, Phys. Lett. A 135, 476 (1989).
- [20] F. Woynarovich and K. Penc, Z. Phys. B 85, 269 (1991).
- [21] C. F. Coll, Phys. Rev. B 9, 2150 (1974).
- [22] A. A. Ovchinnikov, Sov. Phys. JETP 30, 1160 (1970).
- [23] F. Woynarovich, J. Phys. C 15, 85 (1982).
- [24] F. Woynarovich, J. Phys. C 15, 97 (1982).

-
- [25] F. Woynarovich, J. Phys. C 16, 5293 (1983).
- [26] A. Klümper, A. Schadschneider and J. Zittartz, Z. Phys. B 78, 99 (1990).
- [27] T. Deguchi, F. H. L. Essler, F. Göhmann, A. Klümper, V. E. Korepin and K. Kusakabe, Phys. Rep. 331, 197 (2000).
- [28] F. Woynarovich, J. Phys. C 16, 6593 (1983).
- [29] F. H. L. Essler and V. E. Korepin, Phys. Rev. Lett. 72, 908 (1994).
- [30] F. H. L. Essler and V. E. Korepin, Nucl. Phys. B 426, 505 (1994).
- [31] M. Takahashi, Prog. Theo. Phys. 47, 69 (1972).
- [32] M. Takahashi, Prog. Theo. Phys. 52, 103 (1974).
- [33] K.-J.-B. Lee and P. Schlottmann, Phys. Rev. B 38, 11566 (1988).
- [34] N. Kawakami, T. Usuki and A. Okiji, Phys. Lett. A 137, 287 (1989).
- [35] T. Usuki, N. Kawakami and A. Okiji, J. Phys. Soc. Japan 59, 1357 (1990).
- [36] P. D. Sacramento, J. Phys.: Condens. Matter 7, 143 (1995).
- [37] H. Frahm and V. E. Korepin, Phys. Rev. B 42, 10553 (1990).
- [38] H. Frahm and V. E. Korepin, Phys. Rev. B 43, 5653 (1991).
- [39] F. Woynarovich and H. P. Eckle, J. Phys. A 20, L443 (1987).
- [40] F. Woynarovich, J. Phys. A 22 4243 (1989).
- [41] M. Ogata and H. Shiba, Phys. Rev. B 41, 2326 (1990).
- [42] M. Ogata, T. Sugiyama and H. Shiba, Phys. Rev. B 43, 8401 (1991).
- [43] A. Parola and S. Sorella, Phys. Rev. Lett. 64, 1831 (1990);
- [44] A. Parola and S. Sorella, Phys. Rev. B 45, 13156 (1992).
- [45] A. Parola and S. Sorella, Phys. Rev. B 57, 6444 (1997).
- [46] K. Penc, F. Mila and H. Shiba, Phys. Rev. Lett. 75, 894 (1995).
- [47] K. Penc, K. Hallberg, F. Mila and H. Shiba, Phys. Rev. Lett. 77, 1390 (1996).
- [48] K. Penc, K. Hallberg, F. Mila and H. Shiba, Phys. Rev. B 55, 15475 (1997).
- [49] F. Göhmann, A. R. Its and V. E. Korepin, Phys. Lett. A 249, 117 (1998).
- [50] F. Göhmann and V. E. Korepin, Phys. Lett. A 260, 516 (1999).
- [51] E. Jeckelmann, F. Gebhard and F. H. L. Essler, Phys. Rev. Lett. 85, 3910 (2000).
- [52] F. H. L. Essler and A. M. Tsvelik, Phys. Rev. B 65, 115117 (2002).
- [53] D. Controzzi and F. H. L. Essler, Phys. Rev. B 66, 165112 (2002).
- [54] B. Doyon and S. Lukyanov, Nucl. Phys. B 644, 451 (2002).
- [55] F. H. L. Essler and A. M. Tsvelik, Phys. Rev. Lett. 90, 126401 (2003).
- [56] N. M. Bogoliubov and V. E. Korepin, Mod. Phys. Lett. B 1, 349 (1988).

-
- [57] N. M. Bogoliubov and V. E. Korepin, *Int. J. Mod. Phys. B* 3, 427 (1989).
 - [58] N. M. Bogoliubov and V. E. Korepin, *Teor. i Mat. Fiz.* 82, 331 (1990).
 - [59] A. E. Feiguin, and F. Heidrich-Meisner, *Phys. Rev. B* 76, 220508(R) (2007).
 - [60] A. Lüscher, R. M. Noack and A. M. Läuchli, *Phys. Rev. A* 78, 013637 (2008).
 - [61] M. Rizzi, M. Polini, M. A. Cazalilla, M. P. Tosi and R. Fazio, *Phys. Rev. B* 77, 245105 (2008).
 - [62] M. Tezuka and M. Ueda, *Phys. Rev. Lett.* 100, 110403 (2008);
 - [63] M. Tezuka and M. Ueda, *New J. Phys.* 12, 055029 (2010).
 - [64] G. G. Batrouni, M. H. Huntley, V.G. Rousseau and R. T. Scalettar, *Phys. Rev. Lett.* 100, 116405 (2008).
 - [65] M. J. Wolak, V. G. Rousseau, C. Miniatura, B. Gémaud, R. T. Scalettar and G. G. Batroun, *Phys. Rev. A* 82, 013614 (2010).
 - [66] S. Cheng, Y.-Z. Jiang, Y.-C. Yu, M. T. Batchelor and X.-W. Guan, *Nucl. Phys. B* 929, 353 (2018).
 - [67] A. Klümper and R.Z. Bariev, *Nucl. Phys. B* 458, 623 (1996).
 - [68] G. JüKttner, A. Klümper and J. Suzuki, *Nucl. Phys. B* 522, 471 (1998).
 - [69] C. N. Yang and C. P. Yang, *J. Math. Phys.* 10, 1115 (1969).
 - [70] T. Koma, *Prog. Theor. Phys.* 78, 1213 (1987).
 - [71] T. Koma, *Prog. Theor. Phys.* 81, 783 (1989).
 - [72] J. L. Cardy, *J. Phys. A* 17, L385 (1984).
 - [73] J. L. Cardy, *Nucl. Phys. B* 270 186, (1986).
 - [74] H. W. Blöte, J. L. Cardy and M. P. Nightingale, *Phys. Rev. Lett.* 56, 742 (1986).
 - [75] I. Affleck, *Phys. Rev. Lett.* 56, 746 (1986).
 - [76] A. A. Belavin, A. M. Polyakov and A. B. Zamolodchikov, *Nucl. Phys. B* 241, 333 (1984).
 - [77] M. Greiner, O. Mandel, T. Esslinger, T. W. Hänsch, and I. Bloch, *Nature* 415, 39 (2002).
 - [78] R. Jördens, N. Strohmaier, K. Günter, H. Moritz, and T. Esslinger, *Nature* 455, 204 (2008).
 - [79] A. Singha, M. Gibertini, B. Karmakar, S. Yuan, M. Polini, G. Vignale, M. I. Katsnelson, A. Pinczuk, L. N. Pfeiffer, K. W. West and V. Pellegrini, *Science* 332, 1176 (2011).
 - [80] R. A. Hart, P. M. Duarte, T.-L. Yang, X. Liu, T. Paiva, E. Khatami, R. T. Scalettar, N. Trivedi, D. A. Huse and R. G. Hulet, *Nature* 519, 211 (2015).
 - [81] D. Greif, M. F. Parsons, A. Mazurenko, C. S. Chiu, S. Blatt, F. Huber, G. Ji and M. Greiner, *Science* 351, 953 (2016).

-
- [82] M. F. Parsons, A. Mazurenko, C. S. Chiu, G. Ji, D. Greif and M. Greiner, *Science* 353, 1253 (2016).
- [83] L. W. Cheuk, M. A. Nichols, K. R. Lawrence, M. Okan, H. Zhang, E. Khatami, N. Trivedi, T. Paiva, M. Rigol and M. W. Zwierlein, *Science* 353, 1260 (2016).
- [84] M. Boll, T. A. Hilker, G. Salomon, A. Omran, J. Nespolo, L. Pollet, I. Bloch and C. Gross, *Science* 353, 1257 (2016).
- [85] A. Mazurenko, C. S. Chiu, G. Ji, M. F. Parsons, M. Kanász-Nagy, R. Schmidt, F. Grusdt, E. Demler, D. Greif and M. Greiner, *Nature* 545, 462 (2017).
- [86] Y. Kono, T. Sakakibara, C. P. Aoyama, C. Hotta, M. M. Turnbull, C. P. Landee and Y. Takano, *Phys. Rev. Lett.* 114, 037202 (2015).
- [87] C. Krellner, S. Lausberg, A. Steppke, M. Brando, L. Pedrero, H. Pfau, S. Tencé, H. Rosner, F. Steglich and C. Geibel, *New J. Phys.* 13, 103014 (2011).
- [88] V. R. Shaginyan, V. A. Stephanovich, K. G. Popov, E. V. Kirichenko and S. A. Artamonov, *Ann. Phys. (Berlin)* 528, 483 (2016).
- [89] S. Cheng, Y.-C. Yu, M. T. Batchelor and X.-W. Guan, *Phys. Rev. B* 97, 121111 (2018).
- [90] S. Cheng, Y.-C. Yu, M. T. Batchelor and X.-W. Guan, *Phys. Rev. B* 97, 125145 (2018).
- [91] Y.-C. Yu, Y.-Y. Chen, H.-Q. Lin, R. A. Roemer and X.-W. Guan, *Phys. Rev. B* 94, 195129 (2016).
- [92] M. P. A. Fisher, P. B. Weichman, G. Grinstein and D. S. Fisher, *Phys. Rev. B* 40, 546 (1989).
- [93] S. Sachdev, *Quantum Phase Transitions* (Cambridge University Press, Cambridge, 1999).
- [94] Q. Zhou and T.-L. Ho, *Phys. Rev. Lett.* 105, 245702 (2010).
- [95] K. R. A. Hazzard and E. J. Mueller, *Phys. Rev. A* 84, 013604 (2011).
- [96] B. Yang, Y.-Y. Chen, Y.-G. Zheng, H. Sun, H.-N. Dai, X.-W. Guan, Z.-S. Yuan and J.-W. Pan, arXiv:1611.00426, to appear in *Phys. Rev. Lett.*, 2017.
- [97] F. He, Y.-Z. Jiang, Y.-C. Yu, H.-Q. Lin and X.-W. Guan, arXiv:1702.05903.
- [98] A. V. Rozhkov, *Phys. Rev. Lett.* 112, 106403 (2014).
- [99] A. G. Lebed, *Phys. Rev. Lett.* 115, 157001 (2015).
- [100] H. A. Bethe, *Z. Phys.* 71, 205 (1931).
- [101] E. H. Lieb and W. Liniger, *Phys. Rev.* 130, 1605 (1963).
- [102] M. Gaudin, *Phys. Lett. A* 24, 55 (1967).
- [103] P. Fulde, R. A. Ferrell, *Phys. Rev.* 135, A550 (1964).
- [104] A. I. Larkin, Yu. N. Ovchinnikov, *Sov. Phys. JETP* 20, 762 (1965).

-
- [105] Y. Matsuda, H. Shimahara, J. Phys. Soc. Japan 76, 051005 (2007).
 - [106] A. Ptok, K. J. Kapcia, P. Piekarczyk and A. M. Oleś, New J. Phys. 19, 063039 (2017).
 - [107] J. Y. Lee, X.-W. Guan, Nucl. Phys. B 53, 125 (2011) .
 - [108] A. G. Izergin, V. E. Korepin, and N. Yu. Reshetikhin, J. Phys. A, 22, 2615 (1989).

